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Number and Comparison in Early Grade Mathematics: Learning from isiXhosa and English Canonical Texts

INGRID ELIZABETH MOSTERT

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SUPERVISOR: PROF Nicky Roberts

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Declaration

I declare that this is my original research for the purpose of the thesis, *Number and comparison in early grade mathematics: learning from isiXhosa and English canonical texts*, and that all sources I have used or quoted have been indicated and acknowledged by means of complete references.

Ingrid Mostert

30 Sept 2020

Date

Abstract

Teaching early grade mathematics in a rural, no-fee, African language-dominant school, typical of South Africa's mainstream education system, requires different skills and knowledge to teaching early grade mathematics in an urban, fee paying English-dominant school, typical of South Africa's privileged education system. Part of the reason for the widely reported disparity between the two systems that make up South Africa's education landscape is that the knowledge of "what works" is primarily embedded in the privileged education system. The location of this knowledge, together with an implicit assumption that this knowledge can successfully be transfer to and used within the mainstream system, has resulted in a paucity of research that examines the linguistic resources and pedagogical practices of the mainstream system. This is particularly true in relation to teaching and learning mathematics in rural, African language dominant schools. This study contributes to addressing this paucity by undertaking four inter-related studies on the constraints and affordances of teaching mathematics in rural schools in one African language, namely isiXhosa.

In order to examine the constraints and affordances of teaching mathematics in rural isiXhosa-dominant classrooms, four studies, written and published as academic papers, were undertaken. These four papers make up the core of this study. I was the sole author on three of the papers, and first author (supported by my supervisor Nicky Roberts) on one paper titled, 'Diversity of mathematical expression: The language of comparison in English and isiXhosa early grade mathematics texts'. Three of the papers focus on the constraints and affordances of teaching in isiXhosa and one focuses on the constraints and affordances of the pedagogical practice of relying heavily on one teaching source. Collectively the four papers provide an example of the kind of research that can be undertaken by an "outside specialist" who is not proficient in isiXhosa, but who is committed to doing research that is accountable to the majority of South African learners. As a study made up of four papers, there are a variety of different types of research questions. The first is an *overarching* research question which is answered by the collection of the four papers as a whole: 'What research possibilities are available for an "outside specialist" wishing to build the education knowledge project in ways that are accountable to rural isiXhosa-dominant schools?'. The question at the centre of the study is: 'What are the affordances and constraints of teaching early grade mathematics in rural isiXhosa-dominant classrooms, both in relation to the linguistic features of isiXhosa and the instructional practices of teachers?' Each paper answers this central question by providing

specific examples of the affordances and constraints of teaching early grade mathematics in rural isiXhosa-dominant classrooms. In essence, each paper provides an answer to this overarching question by providing an example of the research that is possible for an outside specialist.

In the same way that this study has an overarching research question and a central research question, the study has an overarching methodology and a cross-cutting methodological approach. The overarching methodology was hours of observations of early grade isiXhosa-dominant classes which enabled me to identify relevant questions and units of analysis. The cross-cutting methodological approach was used to study the unit of analysis of each paper, namely the examples of the way a particular mathematical idea is expressed linguistically.

The unit of analysis in each paper was the collection of linguistic expressions of a particular mathematical idea, as they appeared in certain canonical texts, in English and in isiXhosa. The mathematical ideas include number names, comparison, the notion of additive relations and word problems, while the linguistic expressions include words, phrases and sentences used to express a word problems.

In order to analyse the collections of linguistic expressions, the dimensions of variation of each expression were identified and used to set up an example space or typology of linguistic expressions. In order to study language-based differences in how isiXhosa and English express mathematical phrase, the notion of a dimension of variation was extended to include language, thereby bringing together ideas from the field of typology in linguistics and ideas from variation theory. Briefly, the methodological approach begins with collecting examples of linguistic expressions of a particular mathematical idea, as they appear in the canonical texts. These are then studied to identify different dimensions of variation in the collection of examples. Once the different dimensions have been identified the corresponding range of change that is permissible for each dimension is established. For each dimension of variation, the range of permissible change is established and these dimensions and the different values they can take are used to set up the framework for an example space or typology. This framework of an example space is then populated with existing examples and “missing” examples are generated.

The findings from the papers highlight a number of affordances of teaching mathematics in isiXhosa that are not available in English, such as the transparency of number names, the explicit signalling of “the difference” when comparing numbers, and the availability of specialised question words such as *kangakanani* ‘to what extent’. The papers also highlight certain potential constraints such as long number names, a wide variety of ways of expressing “more than” and a lack of exposure to the full range of additive relation word problem types.

By providing specific examples of the implications of teaching mathematics in isiXhosa rather than in English, this study highlights the importance of research into language-based differences in expressing mathematical ideas. For example, by considering the ways in which isiXhosa and English differ in terms of how they express particular mathematical ideas, this study provides insights both into how to leverage isiXhosa when teaching mathematics in isiXhosa dominant early grade classes as well as providing insights into how teachers can support isiXhosa speaking learners who are learning mathematics in English.

Finally, the study also provides an example of how the large corpus of South Africa early grade mathematics texts that have been translated into all nine official African languages can be used as a starting point for teachers, mathematics education specialists and applied linguists to work together to strengthen the mathematics register and to further research the implications of teaching mathematics in African languages. Such research would be embedded in and contribute to the success of the mainstream education system rather than trying to transfer knowledge from the privileged education system.

Dedication

I dedicate this thesis to the Zithulele ooNobalisa team. You welcomed me into your professional and personal lives; you enthusiastically started numeracy clubs even though many of you had negative experiences of mathematics and even though numeracy clubs were not part of your original job description; you graciously allowed me to visit your homes and patiently answered my many questions about how to say things in isiXhosa; and you collected the data that allowed me to confirm hypothesis that I had based on my observations of how you chose to explain mathematical ideas to children. This thesis would not have been possible without you.

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List of abbreviations, acronyms and contextual vocabulary

ANA	Annual National Assessment
B.Ed.	Bachelor of Education
BICS	Basic Interpersonal Communicative Skills
CALP	Cognitive Academic Language Proficiency
CAPS	Curriculum and Assessment Policy Statement
DBE	Department of Basic Education
EGMA	Early Grade Mathematics Assessment
Ex-Model C	State schools reserved for the white population prior to 1994
IMG	Interlinear Morphemic Glossing
NECT	National Education Collaboration Trust
NEEDU	National Education Evaluation and Development Unit
NGO	Non-Governmental Organisation
NMI	Nelson Mandela Institute
PIRLS	Progress in Reading and Literacy Survey
SACMEG	Southern and Eastern Africa Consortium for Monitoring Educational Quality
Transkei	One of the unrecognised states, known as homelands, set up by the apartheid government to promote the policy of separate development for black Africans

Disclaimer

While different races were classified under the apartheid regime and therefore race and the apartheid classifications are referred to in this paper, I acknowledge that there is no biological basis for the notion of race and I view the notion of different racial categories as a political construct. I support the American Association of Physical Anthropologists' statement on race and racism which states that: "Race does not provide an accurate representation of human biological variation. It was never accurate in the past, and it remains inaccurate when referencing contemporary human populations." (American Association of Physical Anthrologogists, 2019, p. 1)

CHAPTER 1: INTRODUCTION

1.1 Scope of thesis

This study consists of four papers, each offering an example of research that can be undertaken by an "outside specialist' aiming to build the education knowledge project in ways that are accountable to the realities of rural, South African, isiXhosa learners and teachers. Each paper is an attempt to understand the constraints and affordances of teaching mathematics in isiXhosa-dominant rural early grade classrooms; both in terms of the linguistic resources of learners and teachers and in terms of prevalent instructional practices. In such classrooms, isiXhosa learners are taught in isiXhosa by a bilingual isiXhosa-English teacher who has isiXhosa as a home language but is not necessarily proficient in the mathematics register of isiXhosa. All four papers have been published in accredited academic journals following a double blind peer review process. I was the sole author on three of the papers. I was the first author on one of the papers (Paper 2 - Diversity of mathematical expression: The language of comparison in English and isiXhosa early grade mathematics texts) where my supervisor Prof Nicky Roberts helped me to polish the analysis and draw the conclusions.

Because of the linguistic limitations I faced, as a white English speaking female, canonical texts rather than spoken language were used as the basis for considering the constraints and affordances of teaching early grade mathematics in isiXhosa. This was done by drawing on early grade canonical mathematics texts including the Curriculum and Assessment Policy Statement (CAPS), the works of the Department of Basic Education (DBE), of the National Education Collaboration Trust (NECT) and of the Nelson Mandela Institute (NMI) as well as a South African version of the Early Grade Mathematics Assessment (EGMA) developed from the international EGMA.

Each paper has as its unit of analysis the linguistic expressions of a particular mathematical idea. These range from numbers, to comparison, to the notion of additive relations. Each paper uses ideas from variation theory to discern the different dimensions of variation that make up the particular linguistic expressions, be they words, phrases or word problems consisting of a problem situation as well as a question. By considering the ways in which each dimension can vary, the papers each set up an example space demonstrating a range of possible linguistic expressions.

For the three papers considering the linguistic resources of learners, the English and isiXhosa versions of these linguistically expressed mathematical ideas were compared. Comparisons were done using a simplified version of interlinear morphemic glossing (IMG), a form of translation used in linguistics in order to present the structure of the source language and not only the meaning. Contrasting the way isiXhosa and English express the same mathematical idea helps to deepen the understanding of the relationship between mathematics and language.

The findings from each paper suggest ways in which the affordances of learning and teaching mathematics in isiXhosa-dominant classrooms can be leveraged; and ways in which the constraints can be mitigated. Many of the findings are also applicable for teachers supporting isiXhosa learners to learn mathematics in English. As a collection, the four papers point to the limited type of research that can be undertaken by someone who is not embedded in rural classrooms and who is not fluent in the language of learning and teaching, in order to help address the scarcity of research accountable to rural learners and teachers.

1.2 Motivation for study

The South African education system has long been described as composed of two very different systems (Fleisch, 2008). The mainstream system serves the majority of learners, typically learners from poor households attending no-fee schools where often no learners speak English at home. In this system many learners do not learn to read for meaning by the end of Grade 3 (Ramadiro & Porteus, 2017). The privileged system serves a minority of learners, typically learners from middle and upper class households, attending fee paying schools where a critical percentage of learners speak English at home and where most learners are able to read for meaning by the end of Grade 3. The learner achievement from these two systems is bimodal with learners from the privileged system performing significantly better than learners from the mainstream system (Taylor, 2011; Shepherd, 2011; Taylor & Yu, 2009; Fleisch, 2008; van der Berg, 2008). These two systems are primarily a legacy of the separate schooling systems set up during apartheid.

While there have been numerous attempts to improve the mainstream education system, even 25 years after the end of apartheid, very little has changed. Ramadiro and Porteus (2017) suggest that part of the reason why the attempts at change have failed is that they have predominantly set out to transfer knowledge from the privileged education system to the mainstream education system. There is an underlying narrative that if teachers and principals in the mainstream system had access to the same knowledge as teachers and principals in the privileged system, then the academic achievement of learners would be the same. However, this is not the case. Instead Ramadiro and Porteus (2017) hypothesise that the problem behind the problem is a lack of appropriate knowledge with in the education system of "what works" in the mainstream system.

The pedagogical knowledge and instructional practices that result in academic achievements in the privileged system do not necessarily provide the same results in the mainstream system. The reason why such knowledge of "what works" is often not transferable is because of the differences between the two schooling systems. Two of the biggest components of classrooms are the instructional practices of the teacher and the linguistic resources of the learners and teachers. The schools in the two systems differ greatly in terms of these two components. In terms of instructional practices, while teachers in the privileged school system typically have smaller classes and are able to provide more individual attention, in the mainstream system learning is largely collectivised as opposed to individualised, chorusing and oral discourses dominate and instructional time is eroded (Hoadley, 2012).

In terms of linguistic resources, in the privileged system the language of learning and teaching is either English or Afrikaans. Teachers are proficient in both the language of instruction and in the mathematics register of that language and, while not all learners speak English (or Afrikaans), a critical percentage of learners do speak English at home, creating the specialised learning conditions for other learners to become proficient in English (Ramadiro & Porteus, 2017). In rural schools in the mainstream system, the language of instruction is an African language, spoken by most, if not all, learners, as well as by the teacher - although often the teacher is not proficient in the mathematics register of that language.

Part of the reason why the relevant knowledge of "what works" in mainstream schools (both urban and rural) has not been developed, is because many early grade mathematics education scholars are embedded in the privileged system, having been taught and having taught in schools within that system. They therefore are proficient in the instructional practices that work in these schools and do not have an understanding of the current instructional practices used in the mainstream system, or the constraints that perpetuate these practices. They also typically do not speak an African language. This is particularly problematic for scholars of early grade mathematics because 70% of learners in the Foundation Phase are taught in an African language in the first four years of formal teaching (Grade R to 4) (Spaull, 2016).

South Africa, like many other countries, has adopted a policy of additive bilingual education (Department of Basic Education, 1997). This policy takes as its starting point that learners are best served by starting school being taught in the language they know best, and that additional languages should be added when learners are ready (Baker, 2001). While there is a wide spread understanding that there needs to be differences in how reading and writing is taught different languages (Pretorius, 2017), a similar understanding of the differences between learning mathematics in English and learning mathematics in isiXhosa is not as wide spread. This is reflected in the process by which canonical texts are produced in languages other than English. Typically, mathematics texts are translated from English into other African languages. Hence, the majority are not being written in a way that takes into consideration the affordances and constraints of teaching mathematics in that particular language.

While all four papers in the study consider the constraints and affordances of teaching mathematics in rural isiXhosa-dominant classrooms, three of them consider the linguistic aspect in particular. These three papers explore the ways in which English and isiXhosa differ in terms of how they express mathematical ideas; and the implications of these differences for learning and teaching mathematics. In the following section I briefly describe the existing literature about the relationship between mathematics and language, and the implications that this relationship has for teaching and for learning mathematics.

1.3 Language and mathematics

The relationship between language and thought has long been a topic of study. While few scholars are proponents of the argument that language determines thought (see Pavlenko, 2011), it is more widely accepted that language influences thought, albeit only habitual thought and not potential thought (Chandler, 2017; Lucy, 1992). The implications of this relationship between language and thought raises questions regarding the relationship between language and mathematical thought. While some scholars consider mathematical objects to have an independent existence, others argue that mathematics does not exist outside of language and that thinking mathematically is a specialised form of using language (Devlin, 2001). In this study the second position is taken.

If mathematics is seen as being created in and through language, it raises the question of whether mathematics would have developed differently if it had been developed in a different family of languages. This question is explored in detail in Barton's (2008) book. A second question that is raised is whether there are language-based differences in how mathematics is conceptualised. There are a number of studies, both theoretical and empirical, that have set out to test whether such language-based differences exist, many of which have attempted to establish the influence of language-based differences of number names on mathematical thinking. An example of language-based differences of number names is seen in a comparison of Indo-European number names and Asian number names. Indo-European number names being less regular and less transparent, compared to Asian number names (Dowker & Li, 2019).

Other studies that have set out to understand the influence of language-based differences on mathematical studies have looked at the mathematical concept of infinity (Kim et al. 2012), and the relationship between the whole and the part, for example, in how fractions are named (Leung, 2017; Bartolini Bussi et al., 2014). While there have been several studies particularly about number names, Lucy (1996, 1997) identifies some common problems with such research, for example, that factors besides language are not controlled for; that conclusions are applied too broadly; and that categories from one language are privileged above those of another language.

This study does not set out to establish the influence of language-based differences on mathematical thought. It does, however, set out to identify language-based differences in how mathematical ideas are expressed linguistically; and to consider the implications of these differences for the learning, and particularly, for the teaching of mathematics. These implications are assessed in both early grade isiXhosa-dominant classes, and other contexts, such as where isiXhosa learners are learning mathematics in an additional language, for example, English.

1.4 Research design

The research design is described by first attending to the research questions – both in relation to the overarching and central research questions which bind the four papers together; and in relation to the specific research questions posed for each paper. This is followed by a short description of the theoretical framework which underpinned the research design. In the theoretical framework I define the unit of analysis, and the lens used to study the unit of analysis. Finally the methodological approaches adopted are described.

1.4.1 Research questions

As a study made up of four papers, there are a variety of different types of research questions. The first is an *overarching* research question which is answered by the collection of the four papers as a whole. The second is a *central* research question, answered in different ways by each individual paper. Finally, each paper has its own set of *specific* research questions.

The overarching question arises from the limitations I face as a researcher who has not been educated or embedded within a rural isiXhosa-dominant school, and who is not proficient in isiXhosa:

> *What research possibilities are available for an "outside specialist" wishing to build the education knowledge project in ways that are accountable to rural isiXhosa-dominant schools?*

In essence, each paper provides an answer to this overarching question by providing an example of the research that is possible for an outside specialist.

The question at the centre of the study is:

> *What are the affordances and constraints of teaching early grade mathematics in rural isiXhosa-dominant classrooms, both in relation to the linguistic features of isiXhosa and the instructional practices of teachers?*

Each paper answers this central question by providing specific examples of the affordances and constraints of teaching early grade mathematics in rural isiXhosa-dominant classrooms. These examples are often in the discussion section of the paper, rather than in the findings sections. This is because each paper has specific research questions related to the specific study undertaken in that paper. These paper-specific questions or aims are listed below.

Paper 1: Number names: Do they count?

- What are the linguistic features of isiXhosa that affect the articulation of numbers in isiXhosa, and how do they affect them?
- What affordances and constraints do these linguistic features provide for the learning and teaching of numbers in early grade mathematics?

Paper 2:[2] Diversity of mathematical expression: The language of comparison in English and isiXhosa early grade mathematics texts

- What comparison phrases are early grade learners expected to be familiar with, according to the English version of canonical South African mathematics texts?
- How do English and isiXhosa comparison phrases differ, as exemplified in canonical South African mathematics texts?

Paper 3: Distribution of additive relation word problems in South African early grade mathematics workbooks[3]

- What additive relation word problem typologies exist in the literature and how can these be combined to form one comprehensive typology?
- What is the prevalence of particular additive relation word problem types in the Foundation Phase mathematics workbooks?

Paper 4: Relative difficulty of early grade compare type word problems: Learning from the case of isiXhosa

- Are certain types of isiXhosa compare type problems easier for learners to solve than others?
- If so, is the relative difficulty influenced by:
    - the formulation of the comparative question?
    - the problem situation?

1.4.2 Theoretical framework

In order to knit the four papers together, an overarching theoretical framework was developed. This consists of two parts: defining the unit of analysis, and defining the lens used to study the unit of analysis.

[2] Note that for this paper I was the first author. My co-author, Nicky Roberts, supported me in the analysis and conclusion of this paper.

[3] Note that in the paper these are stated as aims but for consistency, in this section the aims have been rephrased as questions.

The broad unit of analysis used in this study consists of examples of linguistically expressed mathematical ideas as they appear, in different languages, in early grade South African canonical mathematics texts. Examples from canonical texts are an important unit of analysis because of the central role canonical texts play in the enacted curriculum. This is particularly true in a system where many teachers rely heavily on these texts to determine what they teach and when they teach it. In addition, as these texts have been translated into all eleven official languages, the examples provide a valuable source for comparing the way different languages express the same mathematical ideas.

Each paper has a particular unit of analysis drawn from the study's overarching unit of analysis. In Paper 1 the mathematical idea is numbers and the unit of analysis is *number names* in isiXhosa and English DBE, NECT and NMI workbooks. In Paper 2 and Paper 4 the mathematical idea is comparison. In Paper 2 the unit of analysis is *comparison phrases* in the isiXhosa and English versions of CAPS and the DBE, NECT and NMI workbooks, while in Paper 4 it is *compare type word problems* in the same canonical texts as well as in an early grade mathematics assessment. Finally, the mathematical idea in Paper 3 is additive relations and the unit of analysis is *additive relation word problems* in the English DBE workbooks.

The lens I use to study the examples of linguistically expressed mathematical ideas draws on ideas from variation theory.

The first idea is the notion of a dimension of variation. A central tenet of variation theory is that objects and phenomena have different aspects or dimensions of variation, and that to understand or know an object or phenomenon is to be able to discern the different dimensions of variation of the object or phenomenon. In order to be able to discern these different dimensions, the dimension in question must be varied while all other dimensions or aspects remain constant. For example, to discern the colour of a ball, a child must see other balls with different colours.

The second idea is taken from an extension of variation theory as applied to mathematics education, namely the idea of an example space (Watson & Mason, 2005). The idea of an example space was originally conceptualised as a collection of different types of mathematical examples; ranging from worked examples, to learner exercises, to illustrations of concepts (Watson & Mason, 2005). Watson and Mason (2005) differentiate between personal example spaces and conventional example spaces and suggest that personal example space can be expanded by identifying and then varying each of the dimensions of variation of the objects in the example space.

In this study I use a narrower definition of an example space. The example spaces in this study are collections linguistic expressions of a specific mathematical idea. The example spaces are extended by varying each of the dimensions of a linguistic expression of the mathematical idea in question. In particular, I argue that the language in which a mathematical idea is expressed is an important dimension of variation that is often not taken into account when considering mathematics. Expressing the same mathematical idea linguistically but varying the language used to express it, allows one to see things about mathematics that would otherwise have been difficult to see.

1.4.3 Methodological approaches

In the same way that this study has an overarching research question and a central research question, the study has an overarching methodology and a cross-cutting methodological approach. The overarching methodology was used to identify a relevant research question for each paper. The cross-cutting methodological approach was used to study the unit of analysis of each paper, namely the examples of the way a particular mathematical idea is expressed linguistically.

1.4.3.1 Identifying relevant questions

One of the limitations of scholars who are not embedded in the system they seek to study, is that they are often not in a position to identify relevant research questions. In order to be in a better position to identify relevant questions, I spent many hours observing rural, isiXhosa, early grade mathematics classes and numeracy clubs. This allowed me to observe and hear isiXhosa being used to teach mathematics. During this time, I not only observed isiXhosa being used to teach mathematics, I also asked isiXhosa speakers how they would use isiXhosa to express specific mathematical ideas. For this I used both direct elicitation, asking “How would you say…”, and a more indirect elicitation approach from linguistics where a situation is set up and participants are free to use language in meaningful ways to describe or solve the situation (Edmonds-Wathen, 2019).

My observations of early grade isiXhosa-dominant classes enabled me to identify relevant questions and units of analysis. For the papers focusing on language, observations also led to my engagements with isiXhosa speakers using direct and indirect elicitation approaches.

For example, the focus on number names in Paper 1 was prompted by observing learners count at the beginning of each lesson, observing how long this took when counting in isiXhosa and observing that teachers were not leveraging the transparency of isiXhosa number names to help learners make sense of place value. The focus on comparison in Paper 2 and 4 was prompted by observing how difficult isiXhosa speaking colleagues, friends and strangers found it to translate or construct comparison questions. The focus on additive relation word problems in Paper 3 was prompted by observing a heavy reliance on the DBE learner workbook.

1.4.3.2 Generating example spaces

In each paper I set up or extended an example space or typology for the linguistic expressions of a particular mathematical idea. In order to do this, each paper followed, to a greater or lesser extent, a methodological approach designed for this purpose. Briefly, the methodological approach begins with collecting examples of linguistic expressions of a particular mathematical idea, as they appear in the canonical texts. These are then studied to identify different dimensions of variation in the collection of examples. Once the different dimensions have been identified the corresponding range of change that is permissible for each dimension is established. For each dimension of variation, the range of permissible change is established and these dimensions and the different values they can take are used to set up the framework for an example space or typology. This framework of an example space is then populated with existing examples and "missing" examples are generated. In the case of isiXhosa example spaces, the missing examples were generated by working closely with isiXhosa speakers and isiXhosa scholars. For the papers where the way in which two different languages are compared, a simplified version of interlinear morphemic glossing was used to more clearly present the structure of isiXhosa, for those not fluent in isiXhosa, and to compare the structure of English with that of isiXhosa, at least as it pertains to the linguistic expressions of specific mathematical ideas.

1.5 Papers

In order to give an overall sense of the problem statement and contribution of each paper, each paper's abstract is presented in turn.

1.5.1 Paper 1: Number names: Do they count?

A number of post-colonial countries in Africa have introduced home language instruction for the initial years of schooling. This policy is particularly challenging in mathematics, as there is a paucity of research on how the linguistic features of African languages might be leveraged for the teaching and learning of mathematics.

This paper examines the linguistic features of certain African languages, pertaining to number names, by describing five linguistic features of one particular African language, isiXhosa. The five features are: syntactical category, transparency, regularity, length of words and differences between spoken and written language.

Different early grade mathematics texts in English and isiXhosa are compared to explore the implications of these features for the learning and teaching of mathematics.

The study concludes that there are both constraints and affordances of learning number names in isiXhosa and other related African languages and suggests how the affordances might be leveraged and the constraints mitigated.

Keywords: Agglutinative languages; counting; forward number word sequence; linguistic features; number names

1.5.2 Paper 2: Diversity of mathematical expression: The language of comparison in English and isiXhosa early grade mathematics texts

The comparison of different languages in terms of specific mathematical ideas can provide insights into the nature of mathematics and into the learning and teaching of mathematics in both languages.

This paper uses a typological perspective to contrast the language of comparison of isiXhosa and English, specifically when comparing countable quantities and numbers. Four canonical early grade mathematics texts in South Africa, written in English and translated into isiXhosa, were used as a source and were compared using interlinear morphemic glossing.

An analysis of the texts exemplifies differences between isiXhosa and English. For example, in isiXhosa the word -*ngaphezulu* can mean both ‘more’ and ‘above’. This can cause a conflict when using representations such as the standard hundred chart where bigger numbers appear below smaller numbers. These and other findings may be of interest to instructional designers, teacher educators, and teachers of early grade mathematics in both isiXhosa and English.

Keywords: language of comparison; mathematical language diversity; early grade mathematics; comparison of numbers

1.5.3 Paper 3: Distribution of additive relation word problems in South African early grade mathematics workbooks

Workbooks were introduced by the South African Department of Basic Education (DBE) in 2011. Although the workbooks were designed as supplementary materials, in some schools they are used as the sole teaching text. Therefore, an analysis of the content coverage of the workbooks is warranted.

This article provides such an analysis in terms of additive relation word problems. This article aims firstly to expound on the existing literature to propose a comprehensive additive relation word problem typology and secondly to analyse the prevalence of particular word problem types in the Foundation Phase mathematics workbooks.

This research was conducted in South Africa, focusing on additive relation word problems in Foundation Phase mathematics workbooks. A comprehensive typology of additive relation word problem types was developed based on typologies used in previous studies. All the additive relation word problems in the 2017 Grades 1–3 Foundation Phase mathematics workbooks were categorised according to this typology.

In total, there were 61 single-step additive relation word problems with numerical answers across the three grades. This is a small number in comparison to other countries. There was also an uneven distribution of problem types, with more problems in the easier subcategories and fewer or no problems in the more difficult subcategories.

This article provides evidence for the need to revise the word problems in the DBE workbooks. It also provides a theoretical framework to use in the revision of the workbooks and in any supplementary teaching material developed for teachers.

1.5.4 Paper 4: Relative difficulty of early grade comparison word problems: Learning from the case of isiXhosa

Word problems form an important part of the early grade mathematics curriculum in South Africa. Studies have shown that the relative difficulty of word problems differ: learners are more likely to solve certain types of word problems than others, with compare type problems being the most difficult.

In order to help early grade learners understand and solve compare type problems, it is important to understand the relative difficulty of different types of compare type problems and the factors that contribute to their relative difficulty. While these factors have been studied in English, less research has attended to word problems in other languages, such as isiXhosa.

In this study, a typology of isiXhosa compare type word problems (difference unknown) was set up. The typology included two dimensions, namely the problem situation and the comparative question. The relative difficulties of specific word problems from this typology were compared by analysing the results from an early grade mathematics assessment administered to two cohorts of Grade 1-3 isiXhosa learners in five rural Eastern Cape schools.

The analysis showed that in isiXhosa, as in English, some compare type problems are easier to solve than others. Problems with "matching" situations are easier to solve than problems with "no matching" situations. Problems with alternatively formulated comparison questions, specifically those using *-shota* or *kangakanani*, are easier to solve than those using a more classic formulation.

This study highlights the importance of understanding the ways in which African languages express mathematical ideas in order to identify and leverage affordances for teaching and learning mathematics.

1.6 Overview of main contributions

The study makes two types of contributions. The first consists of the contributions of the individual papers in terms of providing examples of the affordances and constraints of teaching early grade mathematics in isiXhosa-dominant classrooms. The second consists of the contribution of the papers as a whole. These are each discussed below.

1.6.1 Contributions of individual papers

Paper 1 (Number names) identified the regularity and transparency of isiXhosa number names as affordances, as these could be leveraged when teaching place value. The paper also identified the length of number names, as well as the difference between spoken and written number names, as constraints as these make it difficult for learners to work with bigger numbers.

The first contribution of Paper 2 (Comparison phrases) is a typology or example space of comparison phrases, both for English and for isiXhosa. The second contribution is the identification of two affordances in the way isiXhosa expresses comparison, namely that isiXhosa explicitly signals "the difference" with the prefix *-nga* and that the structure of isiXhosa comparison phrases are not affected by the objects being compared. Neither of these affordances are available in English. The third contribution is that the paper identified the fact that *-ngaphezulu* can mean both 'more' and 'above' as a potential constraint when teaching mathematics in isiXhosa, particularly when using representations such as the standard top-down 100 chart where bigger numbers appear below smaller numbers.

The first contribution of Paper 3 (Distribution of word problems) is a comprehensive typology of additive relation word problems, bringing together earlier typologies. This typology was used to make the second contribution, namely establishing that the early grade DBE mathematics workbooks contain relatively few examples of additive relation word problems and that these are not evenly distributed with only three compare type problems and no equalise type problems. This uneven distribution is a constraint in cases where teachers were using the workbooks as primary teaching resource.

Paper 4 (Compare type word problems) provides a typology or example space of compare type (difference unknown) additive relation word problems, both for isiXhosa and for English. Paper 4 established that isiXhosa has two affordances, not available in English, relating to how compare type questions are phrased. Firstly, isiXhosa has a specialised question word, *kangakanani* 'to what extent', which can help reduce the potential confusion between "how many" questions and "how many more" questions. Secondly, in isiXhosa, the loanword *-shota* 'to be short of' can be used in comparison questions in a broader sense than the way 'to be short of' is used in English. Paper 4 empirically tested and confirmed that compare questions using *kangakanani* or *-shota* were easier for learners to solve than compare questions using *ngaphi* and *-zininzi*.

1.6.2 Overall contribution of study

I have identified five areas in which the study as a whole contributes. The first contribution is an example of a methodological approach for studying the implications of teaching mathematics in a language in which the researcher is not proficient. The study exemplifies two main components of such an approach: (1) the need to spend time gaining a functional understanding of the language, as well as observing lessons in which the language is being used to teach mathematics; and (2) the need to work closely with language specialists.

The second contribution is that the study positioned examples from canonical texts, in this case linguistic expressions of mathematical ideas, as a unit of analysis worthy of research. In the four papers the examples from canonical texts were used as a unit of analysis in three ways. They were used (1) as a means to establish the extent to which a learner is likely to encounter a particular mathematical idea; (2) as a starting point for establishing the ways a particular language expresses mathematical ideas; and (3) as a means of comparing the way two languages with different linguistic structures, express the same mathematical idea. The fact that examples from canonical texts could be used in the third way is possible because of the large corpus of mathematical texts that have been translated into all of the eleven official languages of South Africa.

The third contribution is the extension of the idea of a *dimension of variation* (Marton & Booth, 1997) to include *the language used to express a mathematical idea* as one of the dimensions that can be varied when expressing a mathematical idea. Varying the language while keeping all other dimensions fixed, provides a deeper understanding of how different languages express mathematical ideas, and of the relationship between mathematics and language (Edmonds-Wathen, 2019)

The fourth contribution also relates to variation theory, in particular to the notion of example spaces as introduced by Watson and Mason (2005). This fourth contribution is both the example spaces set up in each paper, as well as the methodology developed to set up the example spaces. Each example space is a resource that can be used by teachers and materials designers to ensure that they have considered the full range of ways in which a particular mathematical idea can be expressed mathematically, and to inform their decision with regard to which of these examples learners should be exposed.

The final contribution is that the study as a whole exemplifies how a simplified version of interlinear morphemic glossing can be used to report on how a particular language expresses mathematical ideas, and as a means of comparing the way two different languages express the same mathematical idea.

1.7 Outline of study

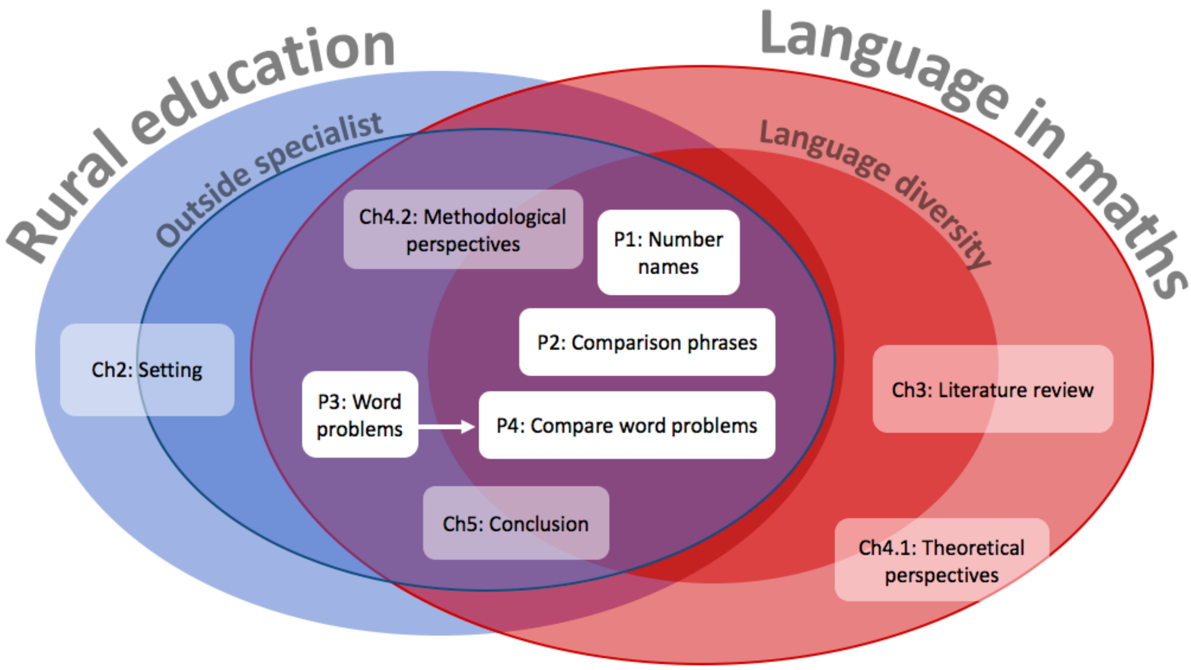


Figure 1.1: Map of study

This study is comprised of three introductory chapters, four academic papers and one concluding chapter. In this section, I locate the four chapters and the four papers in relation to each other; and in relation to the research domain of *language in mathematics*, specifically language diversity, as well as the research domain of *rural education*, as researched by an outsider (see Figure 1.1).

The four papers lie at the heart of the study and are located in the intersection between the research domains of language in mathematics and rural education, as does the final chapter which knits together the four papers.

Chapter 2 sets out the rationale for the study by drawing on rural education research, specifically as it relates to the context of South African education.

Chapter 3 discusses relevant literature regarding language in mathematics, considering both the implications of the close relationship between thought and language for mathematical thinking, and for learning and teaching mathematics in different contexts.

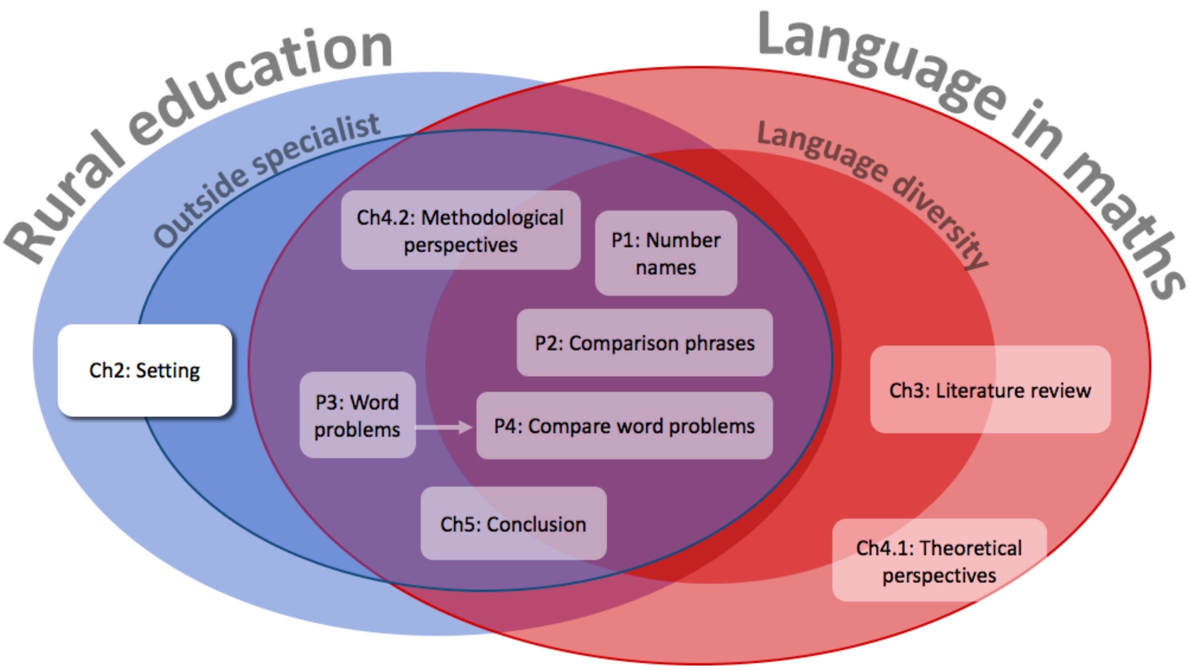
Chapter 4 presents the theoretical framework and methodological approach used in the study. The theoretical framework draws on ideas from variation theory and linguistic typology and is therefore partly located outside the domain of language in mathematics. The methodological approach is comprised of two parts: (1) an overarching approach used to gain an understanding of rural isiXhosa-dominant classrooms; and (2) a cross-cutting analytical approach used, to varying degrees, in each of the four papers, and which lies firmly in the intersection between the domain of language in mathematics and the domain of rural education.

These three introductory chapters are followed by the four papers that make up the core of this study. As can be seen from Figure 1.1, three of these papers (Papers 1, 2 and 4) fall within the domain of language diversity in mathematics education. These papers contrast the ways in which English and isiXhosa express certain mathematical ideas. The remaining paper (Paper 3), while falling within the domain of language in mathematics as it pertains to word problems, does not fall within the domain of language diversity as only English word problems are considered.

Paper 1 (Number names) has been published in *Pythagoras*. It describes the differences between isiXhosa and English number names, as well as the constraints and affordances of each number-naming convention. Paper 2 (Comparison phrases) has been published in *Research in Mathematics Education*. It uses a typological perspective to contrast the language of comparison of isiXhosa and English, specifically when comparing countable quantities and numbers. Paper 3 (Distribution of word problems) was published in the *South African Journal for Childhood Education.* It provides an analysis of the English DBE workbooks in terms of the range of different types of additive relation word problems. While Paper 3 is important because of the over reliance on workbooks within rural schools, it also sets up certain typologies used in Paper 4 (Compare type word problems). Paper 4 has been published in the *African Journal for Mathematics, Science and Technology Education.* It considers the relative difficulties of different types of compare type word problems in isiXhosa, in an early grade mathematics assessment (EGMA) and draws on work from Paper 2 (Comparison phrases) and Paper 3 (Distribution of word problems).

Finally, Chapter 5 synthesizes the findings of the four papers, considers the contributions, limitations and implications of the collection of the four papers and returns to answer the overarching question.

CHAPTER 2: SETTING



2.1 Overview

In this chapter I locate this study within the South African education system, in particular within rural schools, located within the under resourced yet dominant part of the education system. The problem (an education system failing the majority of learners) and the problem behind the problem (an education knowledge project not embedded in, or accountable to the experienced reality of the majority of learners) are described, as well as current solutions that have been tried by the Department of Basic Education (DBE). This chapter also outlines how the current status of the education system, and my position within the system, provide the rationale for the focus and approach of the study.

2.2 The South African education system

The South African education system has repeatedly been described as consisting of two education systems resulting in a bimodal achievement distribution. The description of “two school systems” was first coined by Fleisch (2008) who drew on Mbeki's (1998) description of South Africa as consisting of “two nations” with “two parallel economies”.

In each instance, the origin of the two systems, be it the school systems, nations or economies, can be traced to policies implemented during the apartheid regime – the legacy of which continues even today, more than two decades after it was ended. The fact that this split in the education system arises from apartheid is evident in the lines along which the split between the two systems runs: namely along socio-economic lines; the historical school-system divide (Spaull, 2013) and language lines (Ramadiro & Porteus, 2017). These lines are revealed in numerous studies of achievement results over a number of years using different measures of assessment (e.g. The Southern and Eastern Africa Consortium for Monitoring Educational Quality (SACMEQ), Progress in International Reading Literacy Study (PIRLS), South African Annual National Assessments (ANAs)). For all measures of assessment, South African results produce a bimodal achievement distribution. When the bimodal achievement results are split by former education department (Taylor, 2011), by school language (Shepherd, 2011; Taylor & Yu, 2009) and by socioeconomic status of schools (Fleisch, 2008; van der Berg, 2008), they reveal two normal distributions, one for each of the two systems.

The socioeconomic status of schools is determined by the poverty of the community around the school and is primarily used to determine allocation of funds. Schools are divided into five quintiles, with quintile one being the "poorest" quintile and quintile five the "least poor". Schools in the first three quintiles are no-fee schools and receive most of the government funding for education. While the two education systems can be described in terms of socioeconomic, historic school-system or language differences, they can also be described in terms of the five quintiles. One system serves 80% of children, primarily children from poor or working-class families who attend schools that were previously part of the so-called coloured or black departments of education. Here, the majority of learners, if not all, speak an African language at home, and are taught by teachers who are often not proficient users of English. These are predominantly schools which fall in quintiles one to four. The other system serves the remaining 20% of children, mainly children of middle-class families attending private or "ex-model C"[4] schools where a critical percentage of learners speak English (Taylor

[4] In the late 1980's the Apartheid government proposed four models (A, B, C and D) for restructuring white-only schools. Most schools opted for Model C which was a semi-private structure, with decreased funding from the state, and increased autonomy for schools. Even though the system was discontinued, the term ex-Model C is still used to describe former whites-only government schools.

& Yu, 2009) and teachers are proficient users of English. These are predominantly quintile five schools.

2.2.1 Problem behind the problem

The failure, over more than two decades, of numerous interventions by the state and by non-governmental organisations (NGOs) to bring these two systems closer together speaks to the complexity and systemic nature of the problem. In this section I describe one framing of the problem behind this problem.

A number of education and economics scholars have attempted to describe and account for the ongoing existence of the two education systems. For example, Van der Berg et al. (2016) identified four binding constraints that they consider the reason for the ongoing existence of the two education systems. These are (1) weak institutional functionality, (2) undue union influence, (3) weak teacher knowledge and pedagogical skill, and (4) wasted learning time and insufficient opportunity to learn.

Van der Berg et al. (2016) include both teacher knowledge and pedagogical skills in the third binding constraint, agreeing with Hoadley (2016) that "raising teacher knowledge on its own is unlikely to shift outcomes – teachers need to know how to translate that knowledge for effective learning in the classroom" (p. 32). They emphasise that "content knowledge and pedagogical skill *both* need to be good" (Van der Berg et al., 2016).

In their book *Foundation Phase Matters*, Ramadiro and Porteus (2017) problematise the third of Van der Berg et al.'s (2016) binding constraints and the notion of good content knowledge and pedagogical skill. They argue that the majority of interventions since 1994 that have set out to improve teacher knowledge and pedagogical skills have been based on the implicit assumptions that expertise in the system is concentrated in fee paying schools, that teachers in wealthier schools have stronger content and pedagogical knowledge and that these should be shared with teachers in no-fee schools (Ramadiro & Porteus, 2017). According to this analysis the problem lies with the teachers and principals of no-fee schools.

Ramadiro and Porteus (2017), however, challenge this analysis. They argue that the problem behind the problem lies with the expertise within the system which is aligned to educational context of wealthy schools. In other words, "the research, curriculum, teacher development and policy work informing educational practice is dominated by educationists largely embedded within the educational context of top quintile schools" (Ramadiro & Porteus, 2017, p. 22). Following their argument, the reason why teachers in no-fee schools are not experiencing educational success is not because they do not have access to the knowledge held by teachers in fee paying schools, but because the knowledge within the system (knowledge about what works in wealthy schools) is not relevant for the context of no-fee schools. Essentially, "the 'knowledge project' in education is not well aligned to the social and language resources of the majority of poor and working class children and their teachers" (Ramadiro & Porteus, 2017, p 24). In order to address the problem behind the problem, Ramadiro and Porteus call for efforts to build knowledge and expertise within the South African education system that is aligned to the social and language resources of the majority of schools.

The problem behind the problem identified by Ramadiro and Porteus (2017) is not limited to the knowledge project in education but is also reflected in the perception of the two education systems, as captured in the following quote, still applicable in 2020, from the *Schools that Work* report by the Department of Education (2007):

> In South Africa, the popular conception (or 'hegemonic norm') of schooling is set by [the] privileged sector of [former white] schools. Images of these schools provide 'common sense' notions of what normal South African schooling is (or should be). However, this hegemonic norm is not the numeric norm [i.e., it does not constitute the majority of schools]. Most schools are not like this. These are not typical schools of the mainstream. (p. 29)

Drawing on the *Schools that Work* report and in an attempt to deliberately foreground the realities of the majority of schools over the perceived image of "normal" South African schools, I refer to the education system serving no-fee schools as the "mainstream education system" and the education system serving fee paying schools as the "privileged education system".

At this point, it is important to note that while schools serving poor communities share some similarities, there are important differences in the social and linguist resources available to schools serving poor rural communities and those serving poor urban and peri-urban communities. By their proximity to centres of commerce, learners attending mainstream urban and peri-urban schools are more likely to have access to electricity at home than learners attending mainstream rural schools. Access to electricity also increases the likelihood of access to radio and television and, therefore, to English. Unlike mainstream rural schools, mainstream urban schools are less likely to have linguistically homogenous classes. So while knowledge of what works in mainstream *urban* schools is more likely to be applicable in mainstream *rural* school than knowledge of what works in privileged schools, not all such knowledge will be applicable.

The call for knowledge that is applicable for rural schools is echoed in the literature on rural education. For example, Barter (2008) argues that “alternative epistemological and pedagogical approaches in teaching and learning are required that are aligned to rural education challenges and development” (p. 468-469). Similarly Masinire et al. (2014) argue that “given the complex and culturally diverse contexts of rurality, there is a need for teachers to develop teaching approaches that acknowledge the dynamics and complexity of the rural context” (p. 151).

This study contributes towards addressing the problem behind the problem, specifically as it relates to rural schools. The study sets out to generate knowledge that is relevant for and accountable to the children and teachers in rural schools, particularly as it relates to their linguistic resources. In as much as I, the researcher, am primarily embedded in the privileged education system, this study provides an example of how an outside specialist, such as myself, might work to embed themselves in the mainstream education system.

2.2.2 My position

Before discussing the instructional practices and linguistic resources of rural schools, I turn to consider my own journey and positionality as a mathematics education specialist and how this has influenced the study. My journey before starting this research is relevant as “research, like almost everything else in life, has autobiographical roots” (Seidman, 2006). Similarly, it is important to understand that my life experiences have been informed by viewpoints in relation to research and that these are therefore positional. As explained by Foote and Bartell (2011) a researchers viewpoints are “partial truths, at times oppositional” and that “researchers themselves exist within socio-cultural, political, and historical contexts” (p. 46).

I am a white woman born in the former homeland of the Transkei, one of the unrecognised states set up by the apartheid government to promote the policy of separate development for black Africans (Alexander, 2002). I completed my primary schooling in Mthatha, the capital of the Transkei, during the years just prior to the end of the apartheid regime, and I completed my high school education “outside” the Transkei, after the end of apartheid. Both schools I attended were part of the privileged education system. Even though

I was born and partly educated in a state set up for black South Africans of amaXhosa descent, I was not taught isiXhosa at school and did not learn any more than basic greetings when I was growing up. Even though I learnt some isiXhosa in high school and attended some classes at university, I still only have a limited functional grasp of isiXhosa.

After obtaining an undergraduate degree in mathematics, I taught high school mathematics for a number of years while also completing a master's degree in Education. I subsequently moved into in-service teacher training. Coming from a middle-class family and having attended and taught at ex-model C public schools, I was an specialist embedded in the privileged education system. As I worked with more teachers from mainstream rural schools, I began to question the validity of my role as mathematics education specialist. While I was able to use innovative and engaging approaches to help teachers make sense of mathematics for themselves, I was challenged by the realization that teachers were not able to apply the approaches I was modeling in their own contexts, at least not in sustainable ways.

Even before reading the work of Ramadiro and Porteus (2017), I increasingly felt the need to embed myself in the mainstream education system. I knew that without being embedded in such schools, I would not be able to gain the relevant expertise needed to support teachers within the mainstream education system, the teachers I was supposedly already providing with specialist knowledge. In 2017 I was able to move back to the former Transkei region, now part of the Eastern Cape, to work for an education non-governmental organisation (NGO) and to begin working on this study.

I was fortunate that my role at the NGO allowed me to spend many hours in the Foundation Phase classes of two rural isiXhosa-dominant schools, as well as many hours observing home-based numeracy clubs run by isiXhosa speaking colleagues for isiXhosa-speaking Foundation Phase children. Although I will never be embedded in the mainstream education system to the same extent as those scholars who were taught and who taught in the system, my exposure to these two rural schools and the home-based numeracy clubs has enabled me to better understand this system. In particular it has enabled me to better understand rural teachers' "on-going battle for survival against powerful forces in this field" (Paxton, 2015, p. 201).

2.3 Rural schools

The Department of Basic Education (2015) defines rural education in South Africa as "the provision of quality education in schools in areas with tribal authorities, farming communities and densely populated settlements outside of urban areas" (n.p). According to this definition, the two schools I worked in are rural as they are in an area with tribal authorities. While I use the terms "rural" and "urban", I note that the terms are problematic and difficult to define. As explained in *Emerging Voices: A Report on Education in South African Rural Communities* (Nelson Mandela Foundation, 2005):

> There is no agreement about what constitutes rural and urban areas in South Africa. The movement of people between rural and urban areas makes such definitions all the more difficult. Their meaning and uses also vary considerably depending on who employs them and for what purposes. (p. x)

In South Africa, almost half of schools (11 252 of a total of 25 741) can be considered rural schools (Department of Basic Education, 2015, 2016). According to Gardiner (2008), the majority of these schools are in the Eastern Cape, KwaZulu-Natal and Limpopo provinces. Together these provinces "contain more than half of South Africa's children between the ages of 7 and 18 years (although not all these schools and children are in rural areas)" (Gardiner, 2008, p. 8).

A number of studies have described some of the defining characteristics of rural schools, most of which are applicable to the two schools I worked in (see for example Balfour et al., 2008; Coetzee et al., 2017). These include:

- poor infrastructure,
- large class sizes,
- time spent on things other than teaching,
- long distances that learners, teachers and government officials need to travel to get to schools.

While descriptions of these features are an important starting point, in order to understand rural education it is important to move beyond descriptions towards some kind of theorising. In his thesis on rural school improvement, Paxton (2015) identifies the following as important components of what a theory of rurality should include:

- acknowledgement of the location of rural schools on the margins of the field of play with limited support, choice and voice;
- the very different set of resources available to rural school communities;
- the constraints that space, place and time impose on agents in this field;
- and the interplay between these agents and the forces exerted by the environment they find themselves in. (p. 29)

Of these components, the constraints and the forces exerted on teachers are most relevant to this study.

In the remainder of this section I will discuss two aspects of rural schools, namely the instructional practices typically used by teachers and the linguistic resources that learners and teachers bring to school. While these two aspects of schooling are entwined, they will be discussed separately, particularly because three papers in this study focus on linguistic resources and one focuses on instructional practices. I end this section looking at research in mathematics education in South Africa pertaining to rural schooling.

2.3.1 Instructional practices of teachers in rural schools

In a 2015 review of South African literature describing instructional practices in Foundation Phase classes, Hoadley (2012) synthesises the descriptive features of teaching in South African primary schools. While the studies were not exclusively about rural schools, all of the features were present in the rural classrooms I observed. These include, among others:

- The dominance of oral discourse;
- The privileging of chorusing;
- Slow pacing;
- Erosion of instructional time;
- Learning that is largely collectivised as opposed to individualised; and
- Weak forms of assessment and lack of feedback.

In this section, rather than describe the specific features of the schools I worked in, and the way in which they exemplify the features described in other studies, I will describe some of the "constraints that space, place and time impose on teachers in rural schools" (Paxton, 2015, p. 38). I will do this by reflecting on two of my experiences, both relating to the instructional practices of the teachers in the schools I was working in.

The first experience was realizing, over time, that my initial conceptualisation of the study was unfeasible and inappropriate. The second experience was of team-teaching a lesson on fractions to a Grade 3 class with an isiXhosa-speaking colleague. In the first instance, my observations are made as an outsider. In the second instance, I was temporarily placed as an insider, a teacher experiencing the powerful external forces within a rural classroom.

2.3.1.1 Constraints and forces: An outsider's view

While I was aware of some of the constraints faced by teachers in rural schools, having been involved in in-service teacher training for a number of years, my lack of appreciation of the implications of these constraints is evident in the original conceptualisation of my study. My intention was to use a classroom-based design approach to evaluate the implementation of a ten-day series of lessons in two Foundation Phase classrooms in two rural schools where the language of learning and teaching was isiXhosa. As with other classroom-based design experiments, the plan was for me to teach the 10 lessons in the first iteration, and for the class teacher to take a more active role in the second and third iterations.

I was aware of the constraints that my position as someone not proficient in isiXhosa would present, but it was only after a number of months of classroom observations that I came to understand the constraints that were not due to my position, but due to external forces experienced by the schools and the teachers. I realised that because of these constraints, it would not be possible or appropriate for me to design and implement a classroom-based design experiment in these schools with these teachers at this time.

There were three particular constraints that I observed. The first two constraints relate to the work set out by the provincial department of education. In the Eastern Cape, as of 2018, all Foundation Phase teachers are required to follow the National Education Collaboration Trust (NECT) daily scripted lesson plans. The relationship between teachers and officials from the provincial department of education is primarily one in which the officials check whether the teachers are complying with scheme of work that has been provided. This places two constraints on teachers: when to teach a particular topic, and how long to spend on the topic.

In the light of these two constraints it became clear that it would not be possible for a teacher to assign ten consecutive days for her class to be taught a topic that was not assigned ten days in the NECT lesson plans. In addition, it would have been particularly unreasonable and inappropriate for me to expect a teacher to assign these ten days at a time that was convenient for my study, as was my original plan.

The third constraint relates to instructional practices. As mentioned above, the instructional practices of the teachers I was observing and hoping to work with reflected the instructional practices described in the literature (Hoadley, 2012). I realised that there were two practices that were a central part of the classroom-based design experiment that I had planned that were not part of the teachers' instructional practice. These are the practices of (1) a teacher working with a small group of children while the rest of the class continues with work on their own; and (2) the practice of asking learners to do more than give an answer to a problem, either by asking them to explain how they got their answers or by asking them to generate their own examples. In the light of the literature on the difficulty of changing ones teaching practice (Handal & Herrington, 2003; Lampert, 1988) I decided that it would be unfeasible to expect teachers to adopt these teaching practices.

Initially, when I decided to request that teachers teach a series of lessons far removed from their instructional practice, I framed the constraint primarily in terms of the physical constraint of overcrowded classrooms, and the fact that teachers had not experienced alternative teaching approaches, either in their own schooling or during their teacher training. In framing the constraint in this way I implicitly assumed that if a teacher had access to alterative teaching practices, and that if class sizes were smaller, it would be easy to adopt these alternative approaches.

In fact, I assumed that if I was able to overcome the language barrier through team-teaching with an isiXhosa speaking colleague, that I would be able to employ these alternative approaches. However, when I shifted from an outside observer to an insider who was teaching a lesson, my understanding of the constraint as being located within the teachers' lack of exposure and the overcrowded classroom shifted drastically.

2.3.1.2 Constraints and forces: experienced from the inside

In the second year that I was observing mathematics classes in rural schools, I was asked by one of the teachers that I worked with to teach a lesson on fractions to her Grade 3 class of 67 learners on a day that she was not going to be at school. I agreed and asked an isiXhosa-speaking colleague to team-teach with me. Teaching the class turned out to be a pivotal moment in my journey. I gained many insights from the experience, for example: the extent of learners' poor relationship with fractions, how difficult it is to do group work in a class of more than 60 learners, and how something as simple as making sure all learners have a pencil can take up to ten minutes. But the most important insight I gained was an appreciation for the extent of the forces that "delimit the boundaries of what is possible in schools" (Paxton, 2015, p. 30).

Below is an extract from the notes I made after teaching the lesson. The extract starts at the point where we had handed out a worksheet in which learners had to colour in two fractions on a fraction wall and circle the name of the larger fraction.

> 13 July 2018
>
> Then we try the fraction wall activity.
>
> Mashiya explains. We hand out the papers and the kids start. Some know what to do. Others don't have a clue.
>
> Kids start going to Mashiya to get marked. I have not anticipated this. I haven't even thought about marking the worksheet. I say I'll mark the easy ones[5] and she can explain to the children who don't understand.
>
> **I become like all the teachers.** I mark as quickly as I can. If a child gets everything right I'm so happy. If a kid gets something wrong I quickly try and explain and then send them back to try again and to come stand in the queue again. The queue is always changing, kids push in. Kids push each other all the time. The queue goes into the desks. I become strict. Very strict. Raising my voice and telling kids to stop pushing each other.
>
> The line of children never ends. The kids who don't get it at all are sent to Mashiya. I don't know if they understand that I'm telling them to go and speak to her. There is a crowd of children around her. Not even a line.
>
> At 10:00 I tell Mashiya it's time to stop. The kids keep pushing pages into my hand to be marked. They all want their pages to be marked, even if they get everything wrong.
>
> I don't know what to say to the kids who get everything wrong. I have the excuse that I can't talk to them. What would I say if I could. Where to start?
>
> Noise noise noise all the time. If only the class was smaller. If one could move between desks and mark work at the desk. If kids weren't obsessed with getting their work marked before carrying on.
>
> I feel exhausted. And hopeless.

[5] In other words, the learners who got everything correct

The defining moment in this extract is when I realise that I am acting like all the teachers that I have been observing. I mark without giving feedback, I shout at children who do not behave without giving clear alternative instructions, I feel completely incapable of helping those who do not understand. This is by no means to say that other instructional practices are not possible in mainstream rural schools (as has been demonstrated by the work described by Ramadiro and Porteus (2017)), but rather to say that shifting instructional practices is not merely a matter of a teacher having good intentions or even being exposed to other ways of engaging with learners. The expectations learners bring to school, picked up from parents and older siblings, together with their own experiences of school (which get more ingrained as the learner progresses through more grades) all contribute to how learners behave in class and how teachers react to them.

Again, while this study focuses more on the linguistic resources rather than on the instructional practices of rural schools, this incident makes it clear that without generative instructional practices, no amount of research into the nuances of isiXhosa grammar will make a difference in lessening the pressure of external forces that act on teachers and learners.

2.3.2 Linguistic context

In order to understand the linguistic context of mainstream rural schools, it is necessary to first understand the complexities of language in education in South Africa, both historically and currently, particularly as it pertains to indigenous African languages, such as isiXhosa, and their relationship to English.

2.3.2.1 History of language in education policy

The use of indigenous African languages in formal education in South Africa has a fraught history and continues to be a contentious issue to this day. Until the 1950s, most African learners that enrolled for formal education attended missionary schools. This changed when the predominantly Afrikaans-speaking National Party came to power in 1948. New policies were introduced, based on the belief that different races[6] should develop separately. This

---

[6] As mentioned previously, while different races were classified under the apartheid regime and therefore race and the apartheid classifications are referred to in this paper, the author acknowledges that there is no biological basis for the notion of race. See https://www.physanth.org/about/position-statements/aapa-statement-race-and-racism-2019/

resulted in segregated education, a language policy designed for separate development, and unequal resource distribution (Heugh, 2002). The policies were designed to ensure the value of Afrikaans, and to relegate African languages to lower levels of education and to practical subjects (Mahlalela-Thusi & Heugh, 2002). In particular, the use of African languages in early years was enforced.

Mahlalela-Thusi and Heugh (2002) point out that even though the policies were designed to impose racial segregation, they also resulted in an accelerated development of African languages through the extended use of mother tongue instruction, the development of education terminology and through textbook production. A lot of this work was done by first language speakers of the African languages themselves (Mahlalela-Thusi & Heugh, 2002). It is, however, important to note that the development of African languages was incidental and that the use of African languages in education was not for the cognitive benefit of their speakers (Department of Higher Education and Training, 2015) but rather to limit their access to English, the language of commerce.

2.3.2.2 Current language in education policy

The change in government in 1994 brought with it new language and education policies. The new Language in Education Policy states that learners have the right to be taught in their home language until they are ready to learn in English, or Afrikaans, and that language should not be a barrier for learning (Department of Basic Education, 1997). The policy is based on three tenets. Firstly, that all 11 official languages should be acknowledged, protected and developed, and that the unequal language development of the past should be redressed (Department of Arts and Culture, 2003). Secondly, that the mother tongue (or the language used most proficiently at home) is the most appropriate language of learning. Thirdly, that while learners will need a strong proficiency in at least one other language, usually English, proficiency in new languages should be added to existing proficiencies rather than replacing them.

This third tenet draws on the notion of additive bi/multilingualism. Additive bi/multilingualism (Baker, 2001) occurs when an additional language is built on the foundation of a language that a child already knows. The process involves firstly teaching the child in the language they know best; then introducing (an) additional language(s) as a language of communication. Then, only when a child is ready, the additional language is used in some combination with the home language, as a language of teaching and learning (Ramadiro & Porteus, 2017).

As discussed in more detail in Chapter 3, in describing this process of learning an additional language, Cummins (2000) distinguishes between basic interpersonal communicative skills (BICS) and cognitive academic language proficiency (CALP). Cummins (2000) argues that it takes approximately two years of being exposed to a second language for learners to gain "conversational fluency" in the second language. In contrast, depending on social, linguistic and pedagogical conditions, children take between five and seven years to learn a new language well enough to use it as a primary language of learning (Cummins, 2000).

Even though many studies from bilingual education have demonstrated the problem with introducing an additional language too quickly without sufficient support for the child's home language, as discussed in Heugh (2002), much of this research relies on case studies and small sample sizes and is, arguably, inconclusive (Seid, 2016). This is partly because of the difficulty of conducting large-scale experiments that control for other factors such as socio-economic status, historical disadvantage, geography and the quality of teaching (Taylor & von Fintel, 2016). However, there are two recent studies from the field of economics of education in which large scale longitudinal studies have provided some evidence that educational outcomes are improved if children are taught first in their home language before being taught in an additional language.

The first study made use of a change in national education policy in Ethiopia, allowing schools to determine their medium of instruction. The study showed that education outcomes such as enrolment, and enrolment in the correct grade, improved if learners were taught in their home language (Seid, 2016). The second study used longitudinal data from the South African annual national assessments and showed that home language instruction in the early grades significantly improves the acquisition of English, as measured in Grades 4, 5 and 6 (Taylor & von Fintel, 2016). These two studies support the pedagogic strategies based on the smaller studies from bilingual education research.

2.3.2.3 Language in education in practice

While the current Language in Education Policy emphasises that learners have the right to be taught in their home language until they are ready to learn in an additional language, the policy does not specify when learners are typically ready, which, according to the bilingual education research is five to seven years after first being introduced to the additional language. Unfortunately, although the Department of Education initiated discussions about both curriculum and language policy towards the end of 1995, the discussions were never integrated.

Consequently, in 1997, Curriculum 2005 was announced before the Language in Education Policy and included language as “a separate learning area, and any reference to language beyond the learning area was cursory” (Heugh, 2002, p. 173).

Another consequence of this lack of integration is that the curriculum policy recommends that learners be taught in English from Grade 4 - rather than from Grade 7 or 8 which would be more in line with the additive bi/multilingualism envisaged by the Language in Education Policy. Once again this recommendation impacts the two education systems differently. It is the learners in the already under resourced mainstream education system that are required to navigate the transition from learning in their home language to learning in English[7] when, according to research, they are not yet ready to do so. In contrast, many learners in the privileged system have English as a home language and therefore do not need to make a transition to learning in an additional language.

Additionally, the instructional tools and teacher support systems that are currently available in the system do not provide sufficient support for navigating the complexities of teaching in an African language while simultaneously preparing learners to be taught in an additional language.

Regarding the use of language in teacher training, currently both pre-service and in-service teacher development and support are almost exclusively conducted in English (Ramollo, 2014). Part of the reason for this is that students have been taught mathematics in English since at least Grade 4. Alex et al. (2020) and Roberts and Alex (2020) showed that entry level mathematics education students at a rural university performed better on a mathematics assessment when assessed in English than when assessed in isiXhosa, even though the majority of students spoke isiXhosa at home. This research confirms the assertion by Ramadiro and Porteus (2017) that the education system is failing to provide teachers with the language necessary to teach Foundation Phase mathematics in isiXhosa. This means that “teachers who speak an African language (like isiXhosa) and will teach African-language speaking children (for example in isiXhosa) are trained in English, and then asked to spontaneously back-translate their educational lexicon and learnings” (p. 40).

[7] According to Spaull (2016) 70% of learners are taught in an African language until the end of grade 3, after which the majority are taught in English.

Teachers are not provided with the language, both the technical lexicon and the discourse, to teach in an African language; or to reflect on their teaching in the language they are using to teach (Ramadiro & Porteus, 2017, p. 40). It is encouraging to note that there are two universities that now offer bilingual initial teacher training for Foundation Phase teachers. These are the University of KwaZulu-Natal which offers an isiZulu-English B.Ed. and the University of Fort Hare which offers an isiXhosa-English B.Ed.

2.3.2.4 Language in rural schools

Learners and teachers in rural areas inhabit a very different linguistic landscape, both in terms of English and African languages, than learners and teachers in urban areas, both inside and outside the classroom. Urban areas, particularly big centres of commerce and government, are home to people from all over South Africa and Africa, often resulting in multilingual classes. While in some classes the majority of learners share the same home language with only a few learners speaking other languages at home, in other classes there is no common home language but instead, for example, half the class shares one home language while the other half shares a different home language (Setati, 2012). In contrast, in rural areas far from centres of business and government, the majority of people typically speak one African language and, possibly, some English. In rural schools, almost all learners share the same African home language, and often, particularly in the Foundation Phase, this is also the home language of the teacher.

In terms of access to their home language, while rural children learn to speak their home language before entering school, they typically have not had the opportunity to develop the kind of "school-like" language which is associated with regular access to print material (Ramadiro & Porteus, 2017). In terms of access to English, before and even after entering school, learners in rural areas often have very little access to English. Where learners living in urban areas are likely be exposed to some English through various forms of media, learners in rural areas without access to electricity and the access to media it provides, have almost no exposure to English, except at school. At school this exposure is limited to First Additional Language (English) lessons which are often neglected due to the value placed on Mathematics and Home Language. In many ways English can be described as a foreign language for rural learners (Adler, 2002).

Additionally, while many rural Foundation Phase teachers are bilingual in an African language and English, most are not so-called "balanced" bilinguals (Baker, 2001), with equal control of both their languages. As Ramadiro and Porteus (2017) point out, "While teachers (have to) conduct much of their written professional communication in English and their speech often is peppered with English, they rarely use English-only with colleagues, friends and family, in their homes or communities" (p. 32). Teachers' lack of proficiency in English impacts on their learners' proficiency in English.

Rural learners' very limited access to English means that they are unlikely to develop the basic interpersonal communication skills that are a precursor to developing the cognitive and academic proficiencies needed to learn in English. This is very different to schools in the privileged education system where more children speak English at home. Ramadiro and Porteus (2017) suggest that because of the status of English, once a critical proportion of learners speak English at home, English will be used as a language of communication in the classroom and on the playground. When English is spoken naturally among learners, it creates "specialised learning conditions" which are not available to other learners, particularly not to rural learners (p. 20).

2.3.3 Mathematics research in South Africa

In the previous two sections I have described the instructional practices and linguistic realities of rural schools. While the problem behind the problem, as identified by Ramadiro and Porteus (2017), relates to both mainstream urban and mainstream rural schools, the available mathematics education literature reveals that this problem is particularly true in relation to rural schools.

While there are a number of studies embedded in the realities of mainstream urban schools there are a very limited number of studies that seek to understand and build knowledge accountable to the realities, specifically the linguistic realities, of mainstream rural schools. This inadequate research base, discussed in more detail below, is not unique to South Africa. In America, according to Bush (2005) even though one third of the population lives in rural areas, mathematics education has virtually ignored rural contexts.

Mathematics education is a well-established field of research in South Africa and the pioneering work of Adler (2001, 2002) and Setati (2002, 2005, 2008) in multilingual classrooms is internationally recognised. However, much of the mathematics education research in South Africa focuses on higher grades. Because learners are taught mathematics in English (or in some cases, in Afrikaans) in these grades, it is unsurprising that the majority of research on language in mathematics focuses on bi/multilingual contexts (see for example the review by Setati et al., 2009).

There is however a growing body of work focusing on early grade mathematics both in terms of the realities of rural schools and in terms of language in mathematics. This is in part due to the creation of two numeracy research chairs in 2010. There are however very few studies addressing the use of language in mathematics in early grade rural schools where learners are taught mathematics in their home language which they share with their teacher, even if the teacher is not proficient in the mathematics register of that language.

In terms of studies in mathematics education that are embedded in rural contexts, one example is the work by Maduna (2002) on the use of teaching aids in two rural primary schools. While the study provides insights into the use of teaching aids and recommends that they be used in "under resourced, rural, second language classrooms", the study did not explicitly consider the language of mathematics.

In terms of early grade studies on the language of mathematics, one example is the work being done through the Learning Cognition Laboratory at the University of Johannesburg (Henning & Ragpot, 2015). The Learning Cognition Laboratory studies children learning in different languages, specifically isiZulu, Sesotho, English, and Afrikaans. It aims to address the lack of research in South Africa "pertaining to children's conceptual development and the role of language in the forming of foundational mathematics concepts" (Henning & Ragpot, 2015, p. 2). In their work, the importance of understanding more about language and conceptual development became clear when researchers at the laboratory encountered difficulties when translating an instrument for measuring mathematics cognition.

Another example of language in early grade mathematics is the work by Mdluli (2017) which considers Sepedi as a language of learning and teaching. Mdluli (2017) presents and analyses an excerpt from a Sepedi class from a mainstream urban school. While it is a Sepedi-medium class, Sepedi is only the home language of 31 out of the 42 learners. In the excerpt, the teacher asks the learners to add 10 to 542 and later to subtract 10 from the answer. In her analysis Mdluli (2017) shows how the teacher does not leverage the affordance of the transparency of the Sepedi number naming system.

While there are few studies on language in early grade mathematics, there are even less studies on language in early grade mathematics in rural contexts. One exception is work by

Feza (2016) who studied the basic numeracy skills of isiXhosa Grade R learners and problematised the current language policy that requires the use of the learners' home language as the language of learning and teaching in the Foundation Phase. Feza (2016) argues that "the literacies experienced by students in their early ages of life are different from those perceived by policy makers, researchers and curriculum designers" (p. 576).

Another exception is the work by the Nelson Mandela Institute as described in the book *Foundation Phase Matters* by Ramadiro and Porteus (2017). The NMI worked with a number of rural schools over more than ten years, providing in-school coaching and developing isiXhosa mathematics workbooks which were provided for each learner. The NMI worked closely with teachers over a long period of time and saw significant shifts in instructional practices and learner results on systemic evaluations.

This study adds to the small sample of language-based mathematics education research in rural schools. While the study does not set out to describe the linguistic resources of learners and teachers, nor does it evaluate an intervention aimed to improve learner results, it does add to the research necessary to provide teachers with a "precise technical lexicon for teaching mathematics in isiXhosa" (Ramadiro & Porteus, 2017, p. 40).

2.4 Current solutions

Having described the current state of the South African education system and the instructional practices and linguistic resources of rural schools, I now briefly discuss three wide scale interventions that have been implemented in an attempt improve education in South Africa: These are (1) supplying all learners with DBE workbooks; (2) supplying all teachers with scripted lessons; and (3) the translation of these workbooks and other canonical texts such as the curriculum document from English into the other ten official languages. For each attempted solution I discuss how, according to my observations, these solutions have played out within mainstream rural schools. All discussions and observations are in relation to mathematics, although some might also be true of other subject areas.

2.4.1 DBE workbooks

One of the major achievements of the DBE is the provision, since 2011, of all Grade 1-6 learners in public schools with full colour language, mathematics and life skills workbooks (Department of Basic Education, 2019). Previously, many children would not have had access to any kind of workbook and would only have copied work off the board, thus limiting the kinds of representations and activities available to them. The workbooks were not intended to replace textbooks and were therefore not designed to ensure full curriculum coverage. Instead the workbooks were designed to be a supplementary resource providing "every learner with worksheets to practise the language and numeracy skills they have been taught in class" (Department of Basic Education, 2019). In practice, however, the way in which the workbooks are used again reveals the difference between the two education systems in South Africa.

Mathews, Mdluli, and Ramsingh (2014) describe the manner in which workbooks were used by six Grade 3 teachers, five from mainstream urban schools and one from a mainstream rural school. Their study showed that teachers used the workbooks in disparate ways and that the majority used them in ways that were not in line with DBE's intentions. These observations are similar to my observations of how the workbooks were used in the classrooms I spent time in during 2017. In more than one class, the workbooks became the primary teaching tool. Possible reasons for this are that: while many mainstream schools do not necessarily have enough textbooks, they do have the right number of workbooks in the correct language (NEEDU, 2016); there has been a positive uptake of the workbook by teachers (Taylor, 2013); and curriculum advisors monitor the number of workbook pages completed by learners and marked by teachers.

The use of workbooks as the primary teaching tool in the classes I observed resulted in mathematics lesson that were exclusively determined by the workbook: a teacher would write the examples from the relevant two pages on the board; go through the answers with the whole class; learners would fill in the answers in their book, and then line up to have their work marked. In many cases, this would take the entire 90 minutes allocated to mathematics. In contrast, in the privileged education system where teachers have access to textbooks and other worksheets, anecdotal evidence suggests that the DBE workbooks are typically used only as supplementary materials, often given as homework, and do not determine the scope and sequence of work.

In the schools I observed, the use of the workbooks as a teaching tool meant that many learners only saw the examples and questions that appear in the workbooks, a number of which are flawed. Learners were therefore not exposed to a full range of examples covering the whole curriculum. This limited exposure forms the basis for the third paper in this study and the central role played by the workbooks in determining the enacted curriculum (Porter &

Smithson, 2001) is the reason why the DBE workbooks form part of the unit of analysis in all four papers.

2.4.2 Scripted lessons

In 2013 the National Education Collaboration Trust (NECT) was established by government in response to the challenges facing South African education. The NECT is a partnership between government, business, labour and civil society that aims to pool resources and work together to restore schools and improve educational outcomes in the period ahead.

Arguably, the widest reaching of the NECTs initiatives in the Foundation Phase was the creation of a toolkit for teaching mathematics, home language and first additional language. This toolkit consists of scripted lesson plans (hard copy), learner workbooks (electronic copy) and an assessment framework (hard copy). Unlike the DBE workbooks, the NECT toolkit was intended to be used as a primary teaching resource. The lesson plans, one for each teaching day of the year, were therefore designed to cover the whole curriculum. These daily lesson plans were in many ways a response to the slow pace of teaching (Hoadley, 2012) resulting in many teachers not covering the curriculum in the available time.

A number of design decisions regarding the toolkit and its distribution reveal problematic assumptions about the realities of rural classrooms. The first is the decision to *not* provide each learner with a hard copy of the learner workbook. It can be assumed that the learner workbooks were designed to save time as they include examples from the classwork and homework as they appear in the lesson plan, thus eliminating the need for the teacher to write the examples on the board and to have learners copy the examples into their books. It is also likely, and understandable, that the DBE was not able to print a second workbook for each learner, having already printed the DBE workbooks. However, by not providing each learner with a workbook, the time allocations for certain activities and for assigning homework (5 minutes) are unfeasible, particularly in large classes where most activities take longer than they would in a smaller class. A lack of learner workbooks has also meant that for classwork or homework with representations that learners were unlikely to be able copy from the board (such as number lines), teachers resorted to photocopying the example from the lesson plan, cutting out the representation, pasting the representation on a piece of paper and writing out an isiXhosa translation of the instructions (see Figure 2.1).

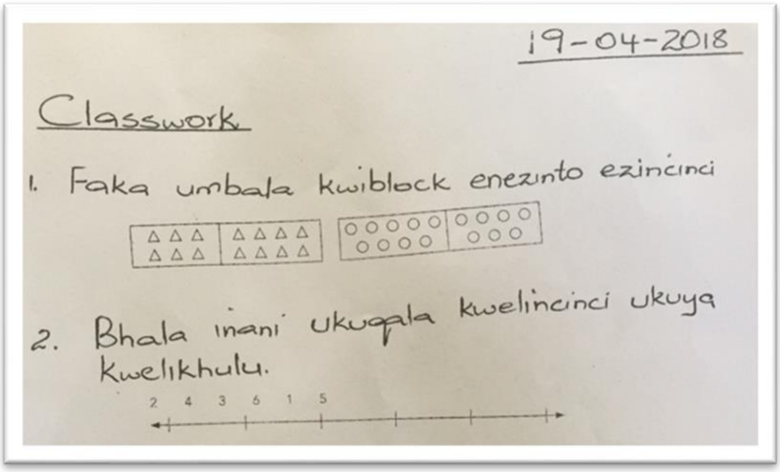


Figure 2.1: Example of classwork worksheet produced by teacher

The second related decision was to provide teachers with only an electronic copy of the learner workbook. With limited computer skills and schools where printing can be a mammoth undertaking, providing teachers with an electronic copy of a document essentially ensures that it will not be used. This is problematic because the learner workbooks were the only component of the toolkit that was translated from English into other languages. Providing teachers with only an English version of the lesson plans suggests an implicit assumption that isiXhosa-speaking teachers trained in English, but teaching isiXhosa-speaking children , will be able to spontaneously translate mathematical phrases and expressions from English into isiXhosa (Ramadiro & Porteus, 2017). While Figure 2.2 demonstrates that teachers did at times translate resources into isiXhosa, this study demonstrates the complexity of the translation process.

The third decision does not directly impact teachers and learners because the learner workbooks are not used, but it does point to an assumption about the linguistic profile of schools. This is the decision to include both English and the language into which the workbooks were translated in the learner workbooks. While having the English and the isiXhosa text together (see Figure 2.2 "Learner workbook") is potentially useful for isiXhosa-English speaking teachers, in a context where learners are still learning to read in their home language, including a second version of the text in a language that is inaccessible to them is likely to hinder rather than to support the learners' understanding of mathematics.

Lesson Plan (hard copy)

Homework

1. Choose two numbers between 21 and 30. Write them down as well as their number name. (Answers will vary.)

2. Write down two numbers that are bigger than 27, but not bigger than 30. (28, 29, 30)

3. Draw 28 beads, showing groups of ten. How many beads are left? (Drawing not shown. 2 groups of 10, 8 left.)

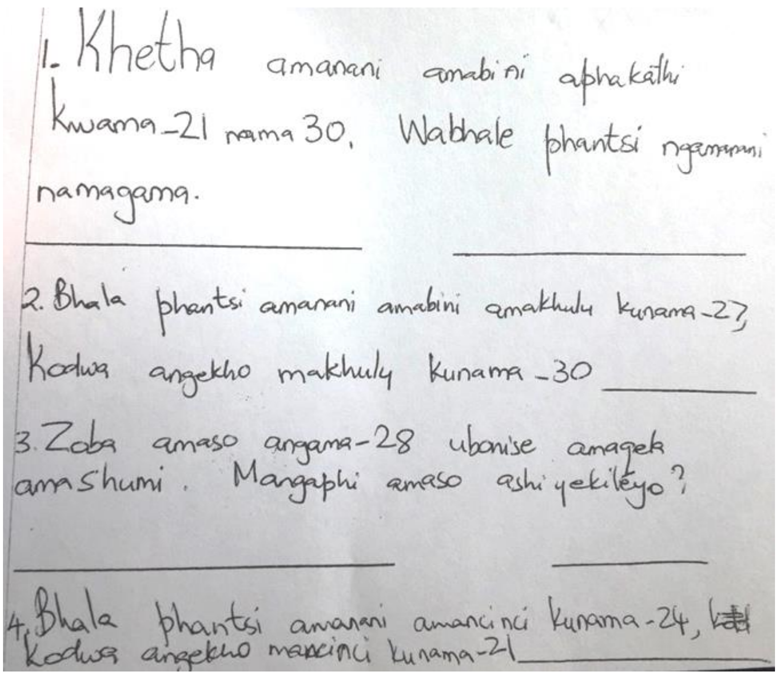
4. Write down two numbers that are smaller than 24, but not smaller than 21. (23, 22, 21)

Learner Workbook (electronic copy)

Homework Umsebenzi wasekhaya

1. Choose two numbers between 21 and 30. Write them down as well as their number name.
   Khetha amanani amabini aphakathi kwama-21 nama-30. Wabhale phantsi ngamanani nangamagama.
2. Write down two numbers that are bigger than 27, but not bigger than 30.
   Bhala phantsi amanani amabini amakhulu kunama-27, kodwa angekho makhulu kunama-30.
3. Draw 28 beads, showing groups of ten. How many beads are left?
   Zoba amaso angama-28 ubonise amaqela amashumi. Mangaphi amaso ashiyekileyo?
4. Write down two numbers that are smaller than 24, but not smaller than 21.
   Bhala phantsi amanani amabini amancinci kunama-24, kodwa angekho mancinane kunama-21.

Created by teacher (hard copy)


1. Khetha amanani amabini aphakathi kwama-21 nama 30. Wabhale phantsi ngamanani namagama.

2. Bhala phantsi amanani amabini amakhulu kunama-27, Kodwa angekho makhulu kunama-30

3. Zoba amaso angama-28 ubonise amaqela amashumi. Mangaphi amaso ashiyekileyo?

4. Bhala phantsi amanani amancinci kunama-24, Kodwa angekho mancinci kunama-21

Figure 2.2: Different versions of a homework task from NECT toolkit

The examples in Figure 2.2 capture some of the complexity of the consequences of the design decisions that were made. Figure 2.2 shows three different versions of the same homework task, one from the NECT lesson plan (in English), one from the NECT learner workbook (in English and isiXhosa) and one created by the teacher (in isiXhosa). While I did not ask the teacher why she created the homework sheet in Figure 2.2, it is likely that it was a combination of a number of factors such as the desire to save time in class (by not having the learners copy down the homework) together with the fact that she either could not print the relevant page from the leaner workbook or that she preferred to have a version with only isiXhosa.

Even though I have highlighted the problems with the NECT learner workbooks as a tool for learners, it should be noted that they do provide a corpus of translated mathematics phrases which can be used to study language diversity in mathematics, as is the case in this study.

2.4.3 Translation of canonical texts

In order to provide quality education for learners in the Foundation Phase in their home language, it is necessary for canonical texts such as the Curriculum and Assessment Policy Statement (CAPS) (Department of Basic Education, 2011a) and government-issued workbooks to be available in all official languages. For a subject such as Home Language, it is clear that curriculum documents and textbooks cannot simply be translated from one language, into another language. Learning to read and write in English is different to learning to read and write in isiXhosa. However, the problem with direct translations in a subject such as mathematics is less clear cut.

An analysis of current translations reveals three overarching problems. The first is use of direct translations that are often out of context (Human Sciences Research Council, 2011). The second is the lack of consistency in how mathematical ideas are expressed in a given African language. This is partly due to the fact that word order in African languages is much more flexible than in English; and the fact that, because isiXhosa does not have a long history of being used to teach mathematics, certain aspects of the mathematics register have not been consolidated. The third is the lack of consideration for language-based differences between the languages and the implications that these have for the teaching of mathematics. For example, it can be assumed that a South African English-speaker given the following: 3×4, would read it as “three groups of four” and that it would therefore be best written as $4 + 4 + 4$. An isiXhosa-speaker, however, would read it as “three repeated four times” and it would therefore be best written as $3 + 3 + 3 + 3$. These three issues point again to the lack of knowledge and expertise in South Africa to provide resources that are relevant for the linguistic resources of the mainstream education system.

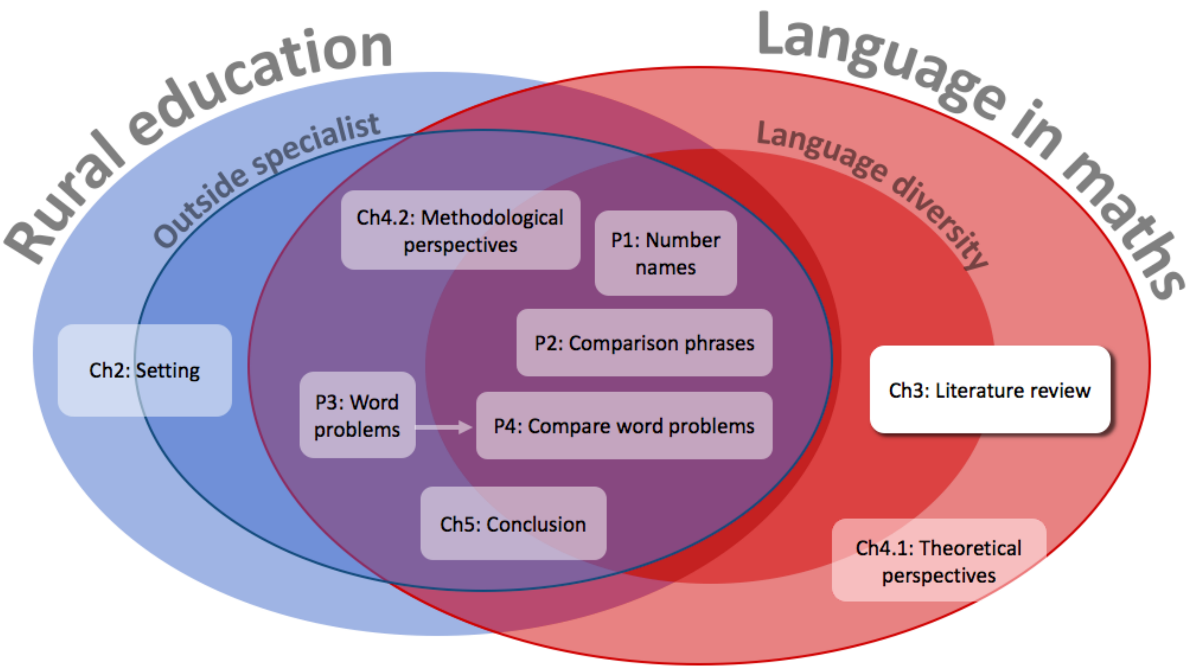
Unfortunately, instead of developing the mathematics register for African languages and taking into consideration language-based differences when translating canonical texts, the lack of time and money within the system perpetuates the current materials development cycle. In this cycle materials are produced in English under immense time pressure and then translated into the remaining official languages, often by translators who do not necessarily have the mathematical knowledge to make informed decisions about what would be the most pedagogically and conceptually useful translation. As pointed out by Gutiérrez (2002), being fluent in a language does not ensure that one is fluent in the mathematics register of that language.

Once again, even though there are flaws in the translation of these canonical texts, they still provide a rich source of examples of how African languages express mathematical ideas and provide a starting point for discussions between teachers, linguistics and mathematics education specialists about the best way to express a mathematical idea, thereby strengthening the mathematics register of African languages. The translated texts also highlight the differences in the linguistic tools used in English and in African languages to express mathematical ideas. Understanding these differences provides insights for teaching and learning mathematics in both English and in African languages, as demonstrated in this study.

2.5 Conclusion

In this introduction I have described the two education systems within the South African education system as well as the problem behind the problem as outlined by Ramadiro and Porteus (2017), namely that the knowledge project is not accountable to the mainstream education system. I have situated my research within a specific part of the mainstream education system within South Africa, namely within rural, isiXhosa-dominant schools. The mainstream education system has been described in relation to instructional practices and linguistic resources. I also considered three broad solutions that the DBE has implemented and how these, according to my observations, have played out in mainstream rural schools.

CHAPTER 3: LITERATURE REVIEW



3.1 Language, thought and mathematics

In Chapter 2 I located this study geographically and socio-politically in rural isiXhosa-dominant classrooms. I also located myself as an "outside specialist" setting out to do research accountable to teachers and learners in rural schools, asking the question "What are the constraints and affordances of teaching mathematics in isiXhosa-dominant classrooms?". In this chapter I locate the study within the mathematics education field of *language in mathematics* and the research domain of *language diversity*.

At the centre of the study lie four separate but related academic papers, each focusing on a different aspect of teaching in mathematics in the context of rural isiXhosa-dominant classrooms, with three papers looking particularly at how isiXhosa expresses mathematical ideas. While each paper includes relevant literature and theory, this chapter covers a wider range and provides a more in-depth exploration of relevant literature than was possible in the academic papers.

In order to frame and guide the discussion of the relevant literature, I consider the following two statements:

- Language influences (rather than determines) thought.
- Mathematics is created in and through language.

I discuss the implications of these two statements, both for mathematical thinking and for the learning and teaching of mathematics. In each case I consider the implications for monolingual contexts as well as for multilingual contexts.

3.2 Language influences (rather than determines) thought

The relationship between language and thought has long been an area of academic interest. Of particular interest is the extent to which language affects thought. This is of interest because while there are many similarities between the grammars of diverse languages, there are also significant differences (Edmonds-Wathen, 2019). If the grammatical structures of one language do not have an exact counterpart in another language, the thinking processes in the two languages could differ (Edmonds-Wathen, 2019).

This perspective is summarised by the so-called Sapir-Whorf hypothesis (Whorf, 1956). The strong version of this hypothesis, also known as linguistic determinism, supposes that language determines and therefore constrains thought. This version has been refuted as too narrow (see Pavlenko, 2011) and has few modern proponents. The weaker, more widely accepted version, also known as linguistic relativism, claims that language only somewhat influences thought. As such, the way we see the world may be influenced by the language we speak (Chandler, 2017). It is important to note that the effects of linguistic relativism relate to habitual thought and not to potential thought (Lucy, 1992). For example, research has shown that speakers of languages without number names are still able to enumerate (Butterworth et al., 2008).

3.2.1 Mathematics is created in and through language

Before discussing the proposition that mathematics is created in and through mathematics, I consider what constitutes mathematics and what constitutes language.

3.2.1.1 Mathematics as universal practices

The term "mathematics" is often taken to refer to the formalised and internationalized discipline of mathematics, what Bishop (1988) calls "Mathematics" ("big-M mathematics"), Barton (2008) calls Near Universal, Conventional mathematics and Edmonds-Wathen (2019) calls school mathematics. There is however a broader view of mathematics. Such a broader view is based on the observations that all peoples, even those not practicing "big-M mathematics", engage with the world mathematically. For example, Barton (2008) points out that every culture has "a system for dealing with quantitative, relational, or spatial aspects of human experience…[a] system that helps us deal with quantity or measurement, or the relationships between things or ideas, or space, shapes or patterns" (p. 10). Bishop (1988) identifies six universal mathematical practices: counting, measuring, locating, designing, playing, and explaining. These systems and practices offer a broader view of what it means to do mathematics and this might differ among different cultural groups, in part due to language differences. While much of this study relates to "school mathematics", when considering the relationship between mathematics and language, overall, a broader view of mathematics is taken.

3.2.1.2 Language as a means of communication

In their overview of mathematics and communication in mathematics education, Morgan et al. (2014) begin their discussion by considering a dictionary definition of language:

> 1. The method of human communication, either spoken or written, consisting of the use of words in a structured and conventional way
>
> 1.1 A non-verbal method of expression or communication, e.g. body language
>
> 2. A system of communication used by a particular country or community
>
> 3. The style of a piece of writing
>
> 3.1 The phraseology and vocabulary of a particular profession, domain or group. (p. 844)

This definition clarifies that language is primarily a means of communication. It is, however, not the only means of communication. While only humans use linguistic means of communication, non-linguistic means of communication, such as facial expressions and gestures, are shared with some non-human species (Santana, 2016).

The definition also indicates that language can be viewed as a system or as the structured use of words. The difference between framing language as structure versus system points to the difference between two theories of language widely used in linguistics and applied linguistics. These are Chomsky's (1965) *Universal Grammar* (language as structure) and (Halliday, 1978) *Systemic Functional Linguistics* (language as system). Halliday and Matthiessen (2014) describe the difference as follows:

> Structure is the syntagmatic ordering in language: "patterns, or regularities, in what *goes together with* what. System, by contrast, is ordering on the other axis: patterns in what *could go instead of* what . . . Any set of alternatives, together with its condition of entry, constitutes a system in the technical sense. (p 22)

The four central claims of Halliday's (1978) work, as discussed by Eggins (2004), are "(1) that language is functional, (2) that its function is to make meanings, (3) that these meanings are influenced by the social and cultural contexts in which they are exchanged and (4) the process of using language is a semiotic process, the process of making meanings by choice". (p. 3)

In general, rather than defining language, research in the field of language and mathematics considers the relationship between language and mathematics and the role(s) of language in learning and teaching mathematics. However, by referring to Halliday's work (for example, Halliday, 1978), particularly his notion of a register, many studies implicitly adopt a functional view of language. In this study a functional view of language is explicitly adopted, particularly in relation to the typological framing discussed in the following chapter.

3.2.1.3 Relationship between mathematics and language

The relationship between mathematics and language is one of the main theoretical issues that divides the field of language and mathematics (Morgan et al., 2014). Some scholars take the position that "mathematical objects have an independent existence, even though they are only experienced through language" (Morgan et al., 2014, p. 845). Other scholars argue that mathematics is entirely dependent on language and does not exist outside of language. For example, Devlin (2001) proposes that the fact that humans are able to create language is the reason why humans are able to do and develop mathematics and that "thinking mathematically is just a specialised form of using our language facility" (p. 4). Similarly, Sfard (2008) argues that mathematics is a "discursive activity" and that mathematical objects are the total of all the different ways of communicating about them.

This second position, which is also the position taken in this study, is summarised by Barton (2012): "we bring mathematics into existence by talking about it" (p. 227). One way of deepening our understanding of the relationship between mathematics and language is to study the mathematical language of diverse languages, in other words, to study the linguistic tools used by diverse languages to express mathematical ideas (Edmonds-Wathen, 2019). This study sets out to do such research.

Having discussed the two original statements, namely that language influences thought and that mathematics is created in and through language, I now turn to consider the implications of these two statements, for mathematical thinking as well as for the learning and teaching of mathematics. Each section first considers the implications for monolingual learners and classrooms, and then for multilingual learners and classrooms.

3.2.2 Implications for mathematical thinking

If language can influence thought and if mathematics is created in and through language, it follows that language can influence mathematical thought. In this section I discuss this implication: first, in terms of language-based differences and the nature of mathematics; then in terms of language-based differences in mathematical thinking; and, finally, in terms of ways in which research on language-based differences can be problematic and ways in which it can be strengthened.

Before discussing these implications, it is important to understand what is meant by language-based differences. Edmonds-Wathen (2019) explains that while English uses specific linguistic features to express certain mathematical ideas or discourses, other languages might have different linguistic resources and might use those resources differently in order to express the same mathematical idea. Examples of differences in linguistic resources and how they are used by different languages to express mathematical ideas are presented in the following sections and in three of the papers that make up this study. In Paper 1 (Number names) the linguistic resources used by isiXhosa and English to express number are compared. In Paper 2 (Comparison phrases) and Paper 4 (Compare type word problems) the linguistic resources used by English and isiXhosa to express comparison are contrasted.

3.2.3 A different formulation of mathematics

The first implication of the influence of language on mathematics relates to mathematics itself. In his book *The language of mathematics: Telling Mathematical Tales,* Barton (2008) argues that because language influences mathematics, if mathematics had been developed in a different family of languages, not in Indo-European languages, mathematics could have evolved differently. For example, Bishop (1988) has identified the “objectifying tendency of mathematics" as one of the defining characteristics of mathematics. Because of this tendency, mathematics often presents processes as if they were things by referring to them with nouns and noun phrases (Schleppegrell, 2007). Barton (2008) argues that this tendency is not due to the inherent "nature" of mathematics but rather to the fact that mathematics was developed in Indo-European languages which tend to make processes into nouns. Other languages such as Mi'kmaq privilege process and verbs over objects and nouns with the words for shapes and numbers acting as verbs (Borden, 2011). Because of this, Barton (2008) argues that it would have been possible to have a "non-objectifying" mathematics, for example a mathematics in which shapes are described in terms of movement through space rather than in terms of objects in space.

3.2.4 Language-based differences in mathematical thinking

A second implication of the influence on language on mathematics is the effect of language-based differences on mathematical thinking. This is captured by the question: "To what extent and in which ways do people who learn mathematics in different languages think about mathematics differently?" There are a number of studies that have set out to answer this by exploring language-based differences in specific areas of mathematical thinking such as number names, infinity and fractions. Leung (2017) distinguishes between two types of studies: theoretical/hypothetical studies in which languages are compared in terms of their linguistic features and hypotheses are made regarding the influence that the differences could have on mathematical thinking; and empirical studies in which these hypotheses are tested. It should be noted that both theoretical and empirical studies of this nature are complex and can easily be flawed. Nonetheless, there are approaches and frameworks from other fields that can strength these types of studies.

3.2.4.1 Number names

Most research into language-based differences in mathematical thinking has focused on number names, particularly on the transparency and regularity of number naming systems. Following in this tradition, the first paper at the centre of this study, "Number names: Do they count?" considers the differences between English and isiXhosa number names.

A number naming system is considered transparent[8] if the number names clearly correspond to the base-10 structure. For example, English is not transparent since the number names for 12 (twelve) and 62 (sixty two) do not give an indication of the number of tens or ones that make up the number. On the other hand, a language such as Chinese is considered transparent: twelve in Chinese is (shi-er), which translates as ten-two; sixty-two in Chinese is (liu-shi-er), which translates as six-ten-two (Dowker & Li, 2019). In both cases it is clear how many tens and how many ones make up the number.

A number naming system is considered regular if, like in Chinese, the same structure is used to name all multiple digit numbers. English does not have a regular counting system for the following reasons: (1) the numbers "eleven" and "twelve" do not give any indication of how many tens or ones the numbers are made up of, (2) the other -teen words are inverted in comparison with the Arabic numerals ('sixteen' first names the ones and then the ten), (3) the -teen words sound similar to the multiples of ten (for example, sixteen and sixty) (Dowker & Li, 2019).

The differences in the level of transparency and regularity of numbers and the influence of these differences on learners' conceptual understanding of number has been the attention of much study. As Dowker and Li (2019) explain: "An oral counting system that is both regular in itself, and transparently related to the written number system may contribute to the precision and accessibility of cognitive representations of number" (p. 2).

The first and widely referenced studies in this domain were done by Miura and her colleagues (Miura, 1987; Miura et al, 1993). In order to assess the influence of transparency of number names on learners' conception of place value, they investigated their representations of tens and units. The learners in the study were six-year-old children from six different nationalities. Three of the languages use transparent number naming systems – Japanese, Korean and Chinese – and the other three use less transparent systems – American English, French and Swedish. The tasks required learners to represent two-digit numbers with base-10 blocks. There were unit blocks and tens blocks with ten segments shown on them. None of the children had used base-10 blocks previously. The studies showed that children using regular

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[8] Note that in many studies transparent number naming systems are also referred to as regular number naming systems.

and transparent counting systems were more likely to represents number with the 10 blocks and the unit blocks. For example, representing 42 with four 10-blocks and two unit blocks. In contrast, the children using less transparent and more irregular counting systems were less likely to use the 10-blocks and more likely to represent 42 with forty-two unit blocks.

Since Miura's original study in 1987, many other similar studies have been conducted comparing different languages and using different means to measure the effect of language-based differences. There have also been studies that have applied these findings, i.e. that transparent and regular number names support learners' understanding of place value. For example, Ejersbo and Misfeldt (2015) looked at Danish number names which are very irregular. They undertook a design research project in which they introduced a second, regular set of number names. The study reported that "the regularity of [the introduced] number names influences the development of number concepts and creates a positive impact on the understanding of the base-10 system" (Ejersbo & Misfeldt, 2015, p. 84). This seems to confirm findings about the positive effect of regular number names.

3.2.4.2 Infinity

Kim et al. (2012) studied language-based differences between English- and Korean-speaking university students' discourses on infinity. They not only hypothesized that there might be language-based differences between English and Korean students, but they also used surveys and interviews to test their hypothesis. The main difference between Korean and English in terms of the word "infinity" is that in Korean, unlike in English, the mathematical word for infinity is not a formalized version of a colloquial term. Instead, in Korean, the mathematical word for infinity is a novel sound inspired by a Chinese term for infinity. The study found that even though the levels of mathematical performance between the two groups were similar, "the English speakers' mathematical discourse was predominantly processual, while the Korean-speaking students' talk on infinity was more structural" (Kim et al., 2012, p. 86). These results support the idea that language-based differences can result in differences in mathematical thinking.

3.2.4.3 Part vs whole

Two theoretical studies considered language-based differences in terms of whether the whole or the part is prioritised. In the first study, Leung (2017) considered the tendency in English, and other Indo-European languages, to move from the specific to the whole (for example, starting with the street number and ending with the country when writing a postal address). He contrasts this with the tendency in Chinese and other Asian languages to move from the whole to the specific (stating the country first and the street number last). Leung (2017) calls for further research on whether these language-based differences will influence learners' conceptualisation of geometric figures, wondering whether "an English student [may] start with the smallest units and view them in the context of the larger units, whereas a Chinese student may start with the larger picture and zoom into the smaller units" (p. 211).

In the second study, Bartolini Bussi et al. (2014) considered the different ways in which languages name fractions. They point out that in European languages, fractions are read top-down, first naming the part and then the whole (three fifths). In contrast, in Asian languages such as Chinese and Burmese, fractions are named, and written, bottom-up, first naming the whole and then the parts (of five parts, take three). Bartolini Bussi et al. (2014) hypothesise that the Asian number naming system might benefit students' conceptual understanding of fractions as students must first split the whole before taking the parts.

3.2.5 Limitations of research on language-based differences

Studies about language-based differences and their influence on mathematical thinking are essentially empirically testing the hypothesis of linguistic relativism. Or, as Saxton and Towse (1998) put it in their description of Miura et al.'s (1993) work, these studies are "an instantiation of linguistic relativity" (p. 68). Such studies are very difficult and often contain a number of common defects, such as privileging the categories of one language or culture over another when doing comparative studies, as set out by Lucy (1997). Lucy (1996) also lists the criteria of studies that do address linguistic relativity such as exploring the extent to which other contextual factors affect the influence of language.

In her paper "Revisiting early research on early language and number names", Moschkovich (2017) uses Lucy's (1996, 1997) defects and criteria to frame her discussion of the early work on number names. She also extends Lucy's (1996, 1997) work to consider other aspects that relate specifically to work in mathematics education.

The following sections draw on Lucy's (1996, 1997) and Moschkovich's (2017) work. Three main criticisms of work on linguistic relativism are discussed: (1) not controlling for or exploring other factors, (2) applying conclusions too broadly, and (3) privileging the categories of one language in comparative studies. For each criticism, examples of studies that have addressed this criticism are also discussed.

3.2.5.1 Not controlling for or exploring other factors

The first criticism is that studies do not control for or explore other factors such as culture and curriculum. In the early studies on language and number names the different groups of learners that took part in the studies came from countries with many differences beyond only language (e.g. Japan, Korea, China, United States of America, France and Sweden).

While Miura and colleagues (Miura, 1987; Miura et al, 1993) did not claim that language was the only factor accounting for differences, later studies have been much more explicit about the other factors that might influence differences. For example, Towse and Saxton (1998) conclude that it is difficult to draw conclusions because there are so many differences, both cultural and educational, between Western and Asian children. Dowker et al. (2008) conclude that "It also suggests that language cannot on its own account for large-scale global cross-national differences in arithmetic" (p. 537). While Dowker and Li (2019) conclude that "The study supports the view that counting systems affect aspects of numerical abilities, but cannot be the full explanation for international differences in mathematics performance" (p. 1).

Later studies have also attempted to address the question of the influence of other factors by ensuring that there is little other than language that differs between the groups being compared. For example, Dowker and colleagues (Dowker et al., 2008; Mark & Dowker, 2015) studied learners exposed to different languages but living in the same country, thereby minimising differences in culture and curriculum. For example, Welsh and English in Wales, Tamil and English in England (Dowker et al., 2008) and Chinese and English in China (Mark & Dowker, 2015). In each case, one of the languages was more transparent than the other. English has a less transparent number naming system than Welsh and Chinese. Tamil is more transparent than English but less transparent than Welsh or Chinese. By controlling for other factors, studies can make much more specific, albeit, more limited claims.

3.2.5.2 Applying conclusions too broadly

The second and related criticism applies specifically to studies on number naming systems. Moschkovich (2017) points out that there is a tendency to apply the results of studies about differences in number naming systems too broadly. For example, a number of studies have included arithmetic tests with the questions relating to the representation of two-digit numbers.

The reasoning is that if learners have a better understanding of place value (because of their language) they will do better in arithmetic tests. The differences in results on these tests have then been used to make a broad conclusion that Asian children's better performance in mathematics is a result of differences in number naming systems. This is problematic, as explained by Moschkovich (2017):

> A jump from a limited claim about early counting to broad causal arguments about the mathematics achievement of Asian children disregards the ecological nature of mathematical reasoning and learning. Extending these studies to more general and broader explanations for differences in mathematics achievement use simplistic views of both culture and language. (p. 4148)

Most studies only support the limited claim that the names for the numbers in Asian languages make the base-ten system more transparent than the number names in non-Asian languages (Moschkovich, 2017). The studies that have included arithmetic tests have shown that the effects of language differences are limited to specific areas of arithmetic (Mark & Dowker, 2015). For example, Welsh children exposed to a regular counting system are better able to read and compare two-digit numbers while Tamil children exposed to a semi-regular counting system perform better on a test of written calculation including a number of sums involving -teens (Dowker et al., 2008).

3.2.5.3 Privileging the categories of one language or culture in comparative studies

The final criticism of comparative studies attempting to address questions of linguistic relativity is that such studies privilege the categories of one language or culture, typically the categories of the "dominant" language (e.g. English) are used to describe the "less dominant" language (e.g. isiXhosa). Instead of privileging one language over another, Edmonds-Wathen (2019) calls for a way to investigate "mathematical concepts and their linguistic expression without preconception of how they will manifest in different languages" (p. 120). In order to do this, Edmonds-Wathen (2019) proposes adopting a typological perspective from linguistics. Typology is concerned with identifying and classifying languages on neutral terms. It is because of this neutrality that Edmonds-Wathen (2019) proposes adopting a typological perspective.

Edmonds-Wathen (2019) demonstrates what adopting a typological approach might look like in her study of English prepositions which is described in Chapter 3. She also identifies two other topics which are of interest in mathematics education and which would be suitable for a typologically-framed investigation. These are the language of modality, used for reasoning, and the language of comparison. The second and fourth papers in this study adopt a typological framing in order to study the language of comparison in English and isiXhosa, in particular, comparison phrases and compare type word problems respectively. Typology, and how it can be used in mathematics education, are discussed in more detail in Chapter 3.

3.2.6 Language elicitation

In the previous sections, examples of studies into language-based differences in mathematical thinking, as well as their defects, were examined. In order to compare the different linguistic tools that languages use to express the same mathematical idea, examples of how languages express mathematical ideas are required. For languages that the researcher is fluent in or languages that have a long history of being used to teach mathematics, gathering such examples is fairly simple. However, for languages that have not been used to teach mathematics, but that nonetheless have a system for referring to the "quantitative, relational, or spatial aspects of human experience" (Barton, 2008, p. 10), generating examples of how such a system is used can be more difficult.

Edmonds-Wathen (2019) suggests that anthropology and linguistics are the two most promising fields of study for methodologies for investigating how different languages express mathematical ideas. Anthropological approaches typically use direct questioning. For example, Barta and Brenner (2009) suggest questions such as "How do you count things in what you do?", "Is sorting/classifying (of objects) used in any way?", and "What shapes are used for various purposes?". Similarly, Borden (2013) used direct questions such as "What is the word for…?" and "Is there a word for …?" in her work with Mi'kmaw students in Nova Scotia.

While these direct questions can prove productive, this is not always the case, particularly when the language being investigated uses different linguistic resources to express a mathematical idea to the resources used in the language spoken by the researcher. When attempting to elicit examples of how to give directions in Yuna, a language from Papua New Guinea, Edmonds-Wathen (2019) describes how research participants gave hypothetical replies and explanations, rather than actual directions. What proved more productive in this setting was the use of structured elicitation tasks from cognitive linguistics. In order to elicit examples of how Yuna could be used to describe the relationship between objects, Edmonds-Wathen (2019) used, among others, the "Man and Tree task" developed by the Cognitive Anthropology Research Group from the Max Planck Institute of Psycholinguistics (Pederson et al., 1998). This task uses a set of cards with different spatial arrangements of a toy man and a tree. In the task, participants are required to describe the relationship between the man and the tree in order to match cards. This task proved productive because participants had a problem to solve and were able to use language in any way to communicate productively.

In this study, direct questions were used to elicit examples of how isiXhosa expresses mathematical ideas. Two additional methods were also engaged, namely the analysis of canonical texts and the observations of isiXhosa-speaking adults engaging with isiXhosa-speaking learners about mathematical concepts, in particular the idea of comparison. These methods are described in detail in Chapter 4: Theoretical and methodological perspectives.

3.2.7 Bilingual thinkers

The previous sections have only considered the implications of the two original propositions from the perspective of monolinguals, asking the question: "What influence does language have on mathematical thinking, assuming the individual speaks and thinks in one particular language?" In most of the studies, the groups being compared were monolinguals exposed to only one language (this was not the case for the Welsh and Tamil learners). However, the implications of the two propositions are not only limited to monolinguals but also have implications for learners speaking and thinking in more than one language, raising the question: "What influence does language have on mathematical thinking when an individual has access to more than one language?".

There is a small but growing field of study looking at students who are able to think and speak about mathematics in two languages. A prime example of this research is the work of Prediger et al. (2019) who consider the different ways in which fractions are named in German and in Turkish and what this might mean for learners who speak both languages. The study draws on the work of Grosjean (2001) who identified different modes of communication available to bilingual speakers including a bilingual mode. In mathematics education Moschkovich (2007), Clarkson (2007) and others have identified the bilingual *complementarity* mode, where learners "use code switching to express complementarity mathematical ideas and provide access to alternate meanings and relationships" (Prediger et al., 2019, p. 192). Prediger et al. (2019) extend this work to include a second component of the bilingual mode, which they will call the bilingual *connection* mode. In this mode speakers use "translanguaging for synthesising new aspects into an internally shared, mentally integrated system" (p. 193).

In their study Prediger et al. (2019) analyse transcripts of discussions between bilingual German-Turkish Grade 7 learners working on fraction problems. The analysis gives examples of the different bilingual modes, i.e. the complementarity and connection mode. They also consider how these differences might be leveraged in learning and teaching, as will be discussed in a subsequent section.

Prediger et al. (2019) call for a reframing of the research on bilingual speakers, from a deficit approach (that suggests that bilingual modes cause interference between languages and that individuals use code switching to compensate for lexical deficiency), to an approach that thinks about bi/multilingual speakers as having access to different ways of thinking about mathematical topics (because the different languages they speak use different linguistic tools to express the same mathematical idea).

3.3 Implications for learning and teaching mathematics

In the previous section I considered the implications of language for mathematical thinking. In this section I consider the implications of language for the learning and teaching of school or "big-M Mathematics". I begin with a discussion of three different perspectives used in mathematics education research to describe the relationship between language and mathematics learning, as identified by Moschkovich (2002). I then consider different language contexts in which children learn school mathematics, with a particular focus on three contexts. The challenges and affordances of learning mathematics in each of these "linguistic contexts" are considered, with special attention given to the role the teacher can play.

3.3.1 Different perspectives on relationship between language and mathematics learning

Since the early work in the 1970s, the number of studies published in the field of language in mathematics has grown considerably. Moschkovich (2002) identifies three perspectives about the learning of mathematics that underlie these studies: learning mathematics as acquiring vocabulary; learning mathematics as the construction of multiple meanings across the everyday and mathematics registers; and, learning mathematics as participation in mathematical Discourse practices. While Moschkovich considered learning mathematics and its relation to language in terms of bilingual learners learning mathematics in a second or additional language, the perspectives are also applicable for monolingual learners. Learning mathematics, even in one's home language, still involves learning to use language in new ways.

Moschkovich's three perspectives will now be described and will then be discussed in terms of the three different linguistic contexts.

3.3.1.1 Acquisition of vocabulary

The first perspective identified by Moschkovich (2002) emphasizes vocabulary and describes learning mathematics as "the acquisition of vocabulary" (p. 191). Research from this view has typically focused on how learners understand individual vocabulary terms or how they translate traditional word problems into mathematical symbols. In this perspective, word problems are taken as a "paradigmatic case" of what it means to learn mathematics (the other being computation) and solving word problems is seen as the prototypical example of how mathematics and language intersect in the classroom (Moschkovich, 2002, p. 192).

Moschkovich (2002) points out that the vocabulary perspective presents a simplified view of language as a list of words that a native speaker knows and, unlike the second perspective, does not consider the multiple meanings of words. Teachers assessing learners' mathematical competence are likely to judge learners based on their knowledge of technical vocabulary, thereby missing other resources learners might use to make meaning of mathematical terms such as "gestures, objects, or everyday experiences" (Moschkovich, 2002, p. 193).

Moschkovich (2002) argues that this perspective is outdated because at present teachers are expecting learners to use many forms of mathematical communication such as "explaining solution processes, describing conjectures, proving conclusions, and presenting arguments" (p. 193). The shift to emphasising communication or "talk" in mathematics classrooms is reflected in the mathematics education literature, for example, Sfard's (2008) work on communication as thinking and, in the case of early grades, Askew's (2016) work on "tasks, tools and talk".

While the vocabulary perspective might be outdated in some classes, in the majority of early grade mathematics classrooms in rural schools in South Africa, it is not yet outdated, as demonstrated by the description, in Chapter 1, of the instructional practices of mathematics teachers. Most teachers still view computation as a paradigmatic case of what it means to do mathematics; they do not expect their learners to explain their thinking or reasoning; and, they view word problems as the only part of mathematics that requires learners to use language.

3.3.1.2 Construction of multiple meanings across different registers

The second perspective addresses some of the problems with the first perspective. It acknowledges that language is more than the sum of the words and morphemes known by a native speaker. It also emphasizes word meanings and uses the notion of registers as introduced by Halliday (1978) who drew on socio-linguistic perspectives of language (de Araujo et al., 2018). Examples of registers include "legal talk" and "baby talk". Unlike the notion of lexicon[9], the notion of register takes into consideration the situation in which a word is used and includes "phonology, morphology, syntax, and semantics as well as non-linguistic behavior" (Moschkovich, 2002, p. 194).

In mathematics education, the register in question is the mathematics register. Halliday (1978) pointed out that counting, measuring, and other "everyday" ways of doing mathematics draw on "everyday" language, but that the kind of mathematics that students need to develop through schooling uses language in "new ways to serve new functions". Halliday (1978) defined the mathematics register as:

> a set of meanings that is appropriate to a particular function of language, together with the words and structures that express these meanings. We can refer to 'mathematics register', in the sense of the meanings that belong to the language of mathematics (the mathematical use of natural language, that is: not mathematics itself), and that a language must express if it is being used for mathematical purposes. (p. 195)

In this perspective, learning mathematics is seen as the construction of multiple meanings across the "everyday" and mathematics registers (Moschkovich, 2002). This requires more than learning new words. As explained by Schleppegrell (2007), "just knowing mathematical words such as more, less, and as many as, for example, is not enough; students also need to learn the language patterns associated with these words and how they construct concepts in mathematics" (p. 143).

This study is primarily located within this second perspective, which is particularly relevant for Paper 2 (Comparison phrases) and Paper 4 (Compare type word problems) as they

[9] "the list of all words and morphemes of a language that is stored in a native speaker's memory; the internalized dictionary" (Finegan & Besnier, 1989, p. 528)

explore the language patterns associated with the words *more* and *more than* in English as well as the equivalent words and language patterns in isiXhosa.

3.3.1.3 Participation in mathematical Discourses

The third perspective emphasises the situated and sociocultural nature of language and mathematics learning, and views learning mathematics as participating in mathematical Discourse practices. Moschkovich (2002) explains that participating in classroom mathematical Discourse practices can be understood as “talking and acting in the ways that mathematically competent people talk and act” which involves more than just the use of technical language (p. 11).

In this third perspective, “Discourse”, as defined by Gee (1996), is distinguished from “discourse”, as typically used in linguistics. In linguistics, “discourse” is defined as “a sequence of sentences that ‘go together’ to constitute a unity, as in conversation, newspaper columns, stories, personal letters, and radio interviews” (Finegan & Besnier, 1989, p. 526). On the other hand, Gee (1996) defines Discourse as “a socially accepted association among ways of using language, other symbolic expressions, and ‘artifacts,’ of thinking, feeling, believing, valuing and acting that can be used to identify oneself as a member of a socially meaningful group or ‘social network’” (p. 131). Applying Gee's (1996) definition to mathematics, mathematical Discourses therefore not only include ways of talking, interacting, reading and writing but also mathematical values and beliefs (Moschkovich, 2002).

Possibly the biggest difference between this third perspective and the previous two perspectives is that it moves away from a deficit view of learners struggling with the differences between the everyday and the mathematics registers or between two national languages, and instead describes the resources and competencies that learners use, from both registers and languages, to communicate mathematically (Moschkovich, 2002). While this study is primarily located in the second perspective, it includes aspects of the third perspective. This is particularly true of the fourth paper which describes a resource used by isiXhosa speakers, namely a loanword from English, to communicate mathematically about the difference between two sets, that is, to communicate about comparison.

3.3.2 Different contexts

As has been established in the previous sections, learning mathematics, no matter what perspective is adopted, involves learning a new language as learners “shift from everyday to more mathematical and precise meanings" (Moschkovich, 2002, p. 194). One important factor in how this learning unfolds is the "linguistic context" of a particular mathematics classroom.

Broadly speaking, it is possible to distinguish between monolingual contexts and multilingual contexts. I define monolingual or linguistically homogenous classrooms as classrooms where:

- monolingual teachers, fluent in the mathematics register of their home language, teach monolingual students, with whom they share a home language.

The situation for multilingual contexts is more complex. As Clarkson (2009) points out, there is no such a thing as *a* multilingual context. Instead there are a wide variety of different multilingual contexts. Clarkson (2004) describes a number of these:

- monolingual teachers teaching a mixture of monolingual and multilingual students;
- monolingual teachers teaching classes of multilingual students, all speaking the same languages;
- multilingual teachers teaching multilingual students in a language different to any of their first languages and
- multilingual teachers and multilingual students who share a language

What Clarkson (2004) does not specify in his description of different multilingual contexts is the relationship that the teacher has with the mathematics register of the language she is using to teach mathematics. It could be argued that by not describing this relationship, it is implied that the teachers in these contexts, be they monolingual or multilingual, are fluent in the mathematics register of the language of instruction in their class. This is not always the case, especially in early grade mathematics classrooms in South Africa. I therefore include an additional multilingual context:

- multilingual teachers teaching emergent multilingual students in a home language they share, but who are not fluent in the mathematics register of the shared home language

This final context is the context that best describes the classrooms informing this study.

For the purposes of this discussion, the different multilingual contexts described by Clarkson (2004) will be considered as one multilingual context, fully acknowledging that there is no *one* context. The discussion will therefore cover three contexts: monolingual, multilingual with teachers fluent in the mathematics register of the language of instruction, multilingual with teachers not fluent in the mathematics register of the language of instruction.

3.3.2.1 Monolingual contexts

The simplest, though by no means simple, context is one in which teacher and learners share the same home language and in which the teacher is not only fluent in the home language but also in the mathematics register of that language. In this section, the linguistic challenges of learning mathematics in this context are discussed, as well as the practices that teachers can use to help learners navigate these challenges. The section ends with a discussion about word problems, which was initially the primary focus of the research concerning the intersection between mathematics and language in the classroom.

Linguistic challenges of learning mathematics

Most of the research about the linguistic challenges of learning mathematics, that is of learning to use the mathematics register and take part in mathematical Discourses, has considered the challenges of learning mathematics in English. In English, challenges include, among others, long dense noun phrases, the grammar of *being* and *having* verbs in relational constructions, precise meanings of conjunctions, and the metaphorical extension of the spatial meaning of prepositions into other areas of mathematics (Edmonds-Wathen, 2019; Schleppegrell, 2007).

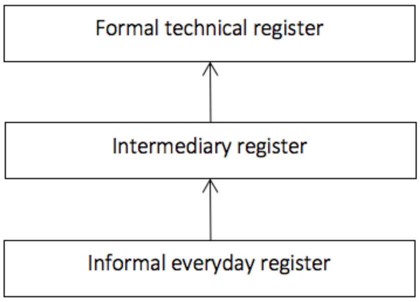
Some of these challenges, such as long dense noun phrases and the precise meaning of conjunctions, do not feature prominently in early grade mathematics. An example of a long dense noun phrase is “the volume of a rectangular prism with sides 8, 10, and 12cm”. However, the challenges of the grammar of *being* and *having* verbs and the way they are used in relational constructions, as well as the challenge of using prepositions in metaphorical ways, do feature in early grade mathematics. For example, one component of learning to count is understanding that numbers come *before* and *after* other numbers. The use of adverbs to talk about the relationship between numbers is particularly complex in isiXhosa, in which the words for the adverbs *before* and *after* (which relate to events in time) are the same words for the prepositions *in front of* and *behind* (which relate to objects in space). In terms of relational constructions, an important concept in early grade mathematics is the relationship between numbers, for example that “eight *is* more than five” or that “three more than five *is* eight”. The lack of standalone *being* and *having* verbs in isiXhosa requires careful consideration when adopting teaching strategies from English.

Even though these challenges exist, Moschkovich (2002) and Schleppegrell (2007) propose ways in which teachers can engage with these challenges. Schleppegrell (2007) points out that teachers can become aware of the linguistic issues in learning and teaching mathematics. Moschkovich (2002) proposes adopting a perspective that, rather than emphasizing the limitations of the everyday register in comparison to the mathematics register, emphasizes the different purposes that the two registers serve, and the ways in which everyday meanings can provide resources for mathematical communication. Teachers can also develop tools for "talking about language in ways that enable them to engage productively with students in constructing mathematics knowledge, some of which will be discussed in the next section" (Schleppegrell, 2007, p. 156).

In the same way that Schleppegrell's (2007) analysis highlights the linguistic challenges of teaching and learning mathematics in English, this study highlights some of the linguistic challenges of teaching and learning mathematics in isiXhosa. There are, however, numerous other linguistic challenges that are beyond the scope of the study.

Supporting learners to moving between registers

Prediger et al. (2016) point out that many mathematics education scholars such as Pimm (1987) and Freudenthal (1991) have called for a teaching strategy that ensures a careful and intentional shift from "everyday" language to "mathematics" language as a means of helping learners make meaning of mathematical concepts and ideas. While some empirical studies have shown that such an approach is effective (e.g. Van den Heuvel-Panhuizen, 2003), many learners still experience serious difficulties in making such a transition. Clarkson (2009) proposes that the reason for the difficulty is related to what an "intermediate register" between the informal everyday register and the formal technical language. This intermediate language is the language often used by teachers in school to explain mathematical ideas and to introduce academic mathematical language. Figure 3.1 shows the relationship between these three registers.



Source: Adapted from Clarkson (2009). Mathematics teaching in Australian multilingual classrooms: Developing an approach to the use of classroom languages. In R. Barwell (Ed.), Multilingualism in Mathematics Classrooms: Global Perspectives (pp. 145–160). Bristol, England: Multilingual Matters.

Figure 3.1: Language use in mathematics learning

While there is generally agreement about the differences between informal and technical mathematical language and about the need for learners to transition from the one to the other, there are very different views on how to support learners in this transition between registers. In particular, views differ in terms of how explicit to make the differences between the different registers. Some views emphasize teaching the language explicitly, for example, Hughes et al. (2016) argue that educators need to "plan for using clear and concise language, which includes identifying inaccurate terms, preferred language, and why changes in language matter" and that learners should be exposed to "clear, concise, and uniformly used language" (p. 4).

Similarly, Schleppegrell (2007) argues that by explicitly focusing students' attention on linguistic features it is possible to help students understand technical mathematical terms. She draws on the work of Pimm (1987) suggesting that one way to develop learners' mathematics register is by having learners talk about mathematics as they solve problems, encouraging them to talk formally. For example, by identifying what "it" and "this one" refer to and by referring to numbers or symbols instead of the pronouns, which typically happens when everyone can see the work on the blackboard (Schleppegrell, 2007).

While also proposing that learners should engage with and talk about the mathematics in order to learn how to use the mathematics register, those who take a more socio-linguistic view recommend that teachers use less direct instruction and rather use strategies such as revoicing. When using revoicing, instead of the teacher pointing out and correcting the mistake in the language used by a learner, the teacher simply repeats what the learner has said but uses the correct mathematics register. This ensures that the meaning of what the learner has said is emphasized, rather than the form of what she has said. This is demonstrated in the following example from Moschkovich (1999):

> Student: “The rectangle has par ... parallelogram... and the triangle does not have parallelogram.”
>
> Teacher: “He says that this [a triangle] is not a parallelogram”

By revoicing, the teacher shows that the learners’ contribution is accepted while at the same time keeping the discussion mathematical.

Deciding on how explicit to make the correct use of technical mathematics language, is a tension described by Adler (1999) as the “dilemma of transparency”. She explains this dilemma as follows:

> On the one side, that explicit mathematics language teaching, in which teachers attend to pupils’ verbal expressions as a public resource for class teaching, appears to be a primary condition for access to mathematics, particular for pupils whose main language is not the language of instruction. On the other side, however, there is always the possibility in explicit language teaching of focusing too much on what is said and how it is said. (p. 48)

This tension is described in more detail in the section on “multilingual contexts”.

While strategies for helping learners use the mathematics register are important, they are only useful in as much as the teachers themselves are fluent in the mathematics register. In the classrooms that this study is embedded in, many teachers were not fluent in certain aspects of the isiXhosa mathematics register, either because they had not been taught mathematics in isiXhosa, or because they received their teacher training in English, as described in Chapter 1. By describing how isiXhosa canonical texts express comparison phrases and questions, Paper 2 and Paper 4 in this study contribute to strengthening the isiXhosa mathematics register, thereby helping to lay the foundation for more relevant teacher training to be delivered (Setati, 2012).

One component of the mathematics register is the correct formulation of word problems. Word problems and their relationship to language in mathematics are discussed in the following section.

Word problems

In early grade word problems, everyday language is used to refer to mathematical concepts, for example in the problem “Sbu has 6 apples, he eats 2, how many does he have left?” the everyday action of eating can be linked to the mathematical concept of subtraction. Young children are able to solve additive relation word problems before they have been taught addition and subtraction (Verschaffel & De Corte, 1993). It is therefore possible to use word problems to help learners make meaning of mathematical concepts, rather than viewing word problems as an opportunity for learners to "apply" calculation strategies. This approach is demonstrated in the realistic mathematics education tradition developed in the Netherlands (van den Heuvel-Panhuizen, 2000).

If word problems are viewed as an opportunity for learners to use everyday language to make meaning of mathematical concepts, it is important that learners have enough opportunities to engage with word problems. Even though teachers in rural schools typically do not view word problems in this way, Paper 3 (Distribution of word problems) in this study examines the number and variety of word problems learners encounter in the workbooks provided by the Department of Basic Education (DBE).

As mentioned previously, word problems were originally seen as the "prototypical" example of the intersection of language and mathematics. For this reason, there is much research on word problems that focuses on the language used in word problems and the influence that this language has on the difficulty of word problems. In terms of word problems presented in the early grades, a number of studies have demonstrated how word problems used in early grades can be modified to make the problems easier for learners to solve (see for example Carpenter et al., 1981; Hudson, 1980; Lindvall & Ibarra, 1980). These studies are discussed in detail in Paper 4 (Compare type word problems), which tests whether modifications to isiXhosa word problems also result in problems that are easier for learners to solve.

Similar studies have been done for word problems used in higher grades. For example, Abedi and Lord (2001) compared "standard" and "modified" word problems and found that learners typically preferred modified word problems. Ways in which the word problems in their study were modified included replacing passive verb forms with active verb forms, shortening the length of long nominal clauses, and replacing conditional clauses with a separate sentence.

3.3.2.2 Multilingual contexts with teachers fluent in mathematics register

In the previous section, the linguistic challenges of learning mathematics were discussed for "monolingual" contexts, or what Barwell et al. (2019) refer to as "linguistically homogenous" classrooms. I now consider multilingual or "linguistically diverse" classrooms. As mentioned previously, it is problematic to refer to "a multilingual context" as there are very different configurations of multilingual contexts with either teachers or learners or both having access to more than one language, and with teachers and learners sharing or not sharing a common language, be it a home language or another language.

Barwell et al. (2007) argue that fewer mathematics classrooms are now entirely linguistically homogenous as a result of two international developments. The first is the increased movement of populations globally as more migrants, for various reasons, move from one cultural environment to another. Due to this increased movement, there are significant increases in the numbers of children expected to learn mathematics in world languages (like English, or Spanish), when their home or main language is not a world language. In relation to teaching mathematics in English, this has resulted in research concerning "English language learners", and focusing on children's lack of English fluency for mathematics learning, and how to better support their learning of mathematics in English.

The second international development is the growth in minority and indigenous peoples' movements which advocate for learning mathematics in immersion and bilingual settings (Barwell et al., 2007). In this regard, there is a growing body of research that focuses on indigenous languages as a resource, and on the different ways that mathematical ideas and processes are expressed in different languages (Edmonds-Wathen, 2019). Such research relies on the comparison of different languages, in particular, the comparison of languages that are structurally different (Barton, 2008).

These two developments have resulted in an increase in the number of non-linguistically homogenous classrooms and, together with this, an increase in research about the use and influence of language in these settings. This is evident from the extent of the reviews on relevant literature, for example (Barwell et al., 2007, 2016; de Araujo et al., 2018; Planas & Schütte, 2018).

From this wide range of literature, I will focus on the additional challenges, and the affordances, faced by and available to learners learning mathematics in a language different to their main language, as well as discussing some strategies teachers might use to support learners in moving between registers.

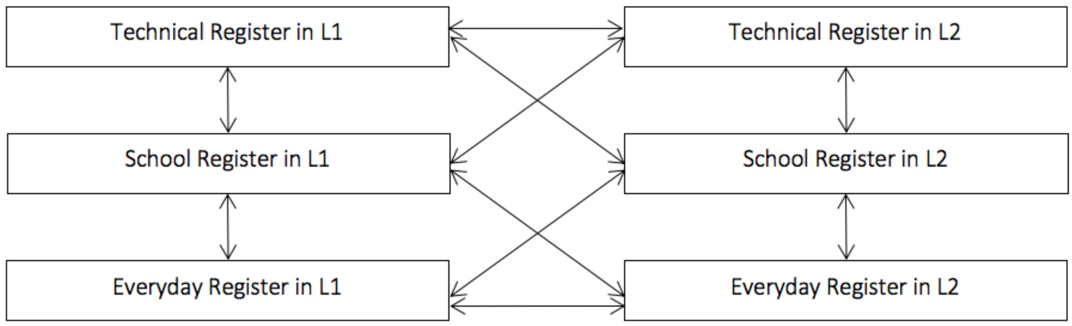
Further linguistic challenges of learning mathematics in an additional language

For learners learning mathematics in an additional language, all the differences between everyday language and the mathematics register are even more difficult to navigate as the learners are learning everyday language and the mathematics register at the same time. These learners not only have to learn the mathematical concepts, but also the language in which these concepts are embedded (Essien, 2018). Or, as Miller and Warren (2014) explain:

> While for all students, the language of mathematics can be difficult to understand; for [English as second language] ESL students, it is far removed from their everyday speech. Hence, for these students, their difficulties are amplified: the language of instruction is unfamiliar, and the language of mathematics is foreign. (p. 794)

The challenges faced by learners learning mathematics in an additional language are captured by the work of Cummins (2000) on different kinds of language proficiency. Cummins (2000) distinguishes "cognitive academic language proficiency" (CALP) from "conversational language proficiency" or "basic interpersonal communicative skills" (BICS). While children are able to learn a second language well enough to communicate in everyday situations within two years, the international consensus is that they require between five and seven years to master the educational language competencies that are needed for learners to succeed in the classroom (Cummins, 2000).

The different registers and different languages that multilingual learners move between are represented in Clarkson's (2009) extended model (Figure 3.2). In this model, Cummins' (2000) notion of BICS is located in the everyday register, while his notion of CALP is located within the academic mathematics register.



Source: Adapted from Clarkson (2009). Mathematics teaching in Australian multilingual classrooms: Developing an approach to the use of classroom languages. In R. Barwell (Ed.), Multilingualism in Mathematics Classrooms: Global Perspectives (pp. 145–160). Bristol, England: Multilingual Matters.

Figure 3.2: Three-tiered model of registers

Clarkson (2009) emphasises that learners should be able to move flexibly between different registers. He argues that learners should be able to bring their own lived experiences to their learning of mathematics, and that these lived experiences are likely to be encoded in the everyday language of their first or main language. He also points out that some cells might be empty, for example, if the learner's first language has not been used to teach school mathematics, the top left-hand cell would be empty (Clarkson, 2009).

It is important to note that if the home language of an English language learner is linguistically remote from the language of instruction, there are additional challenges faced by the learner, over and above unfamiliarity with the new language. For example, there may be no direct equivalent term in the learner's home language that can be used to clarify the meaning of a particular word (Jones, 1982), as is the case with *middle* which does not have an equivalent term in Mi'kmaq (Borden, 2013). Another challenge is that learners might bring understandings of words from their main language and use them to make (incorrect) sense of the technical term in the language of instruction, such as in the study by Kazima (2007) on Chichewa-speaking students' understanding of probability vocabulary, in which many learners indicated that *certain* means *unlikely* - which Kazima hypothesized was because the Chichewa word for *certain* (as in 'a certain book') translates into *linalake*, literally 'one of many' (p. 185). Finally, the linguistic tools used to construct a mathematical idea in the language of instruction might be different to the tools used in their main language, as in Pitjantjatjara which uses the case system rather than prepositions to indicate the relationship between objects (Edmonds-Wathen, 2019).

Further strategies for supporting additional language learners

While all mathematics teachers are faced with the challenge of helping learners move between the everyday register and the academic mathematics register, Barwell (2009) points out that in multilingual classrooms with learners whose home language is not the language of instruction and who are not yet proficient in that language, teachers have to balance their attention between mathematics, the language of instruction and the mathematical language. This section discusses some additional strategies, as described in the literature, for teachers who have to balance the attention given to these three important aspects of learning mathematics.

In describing the challenges faced by learners learning mathematics in a language different to their main or home language, as was done in the previous section, it is easy to take a deficit view of these learners and, for example, to propose approaches such as using simpler vocabulary and words (Fillmore, 1982). However, this approach does not necessarily result in learners having a better understanding of the mathematical ideas. Instead, learning can be hindered by over-simplifying a rich mathematical concept (Miller & Warren, 2014).

There are, however, other approaches that do not take the deficit perspective, that have been shown to be helpful in supporting learners who are learning mathematics in a language different to their home language. Because this study is embedded in classrooms where learners are taught in their home language, only a selection of these approaches will be discussed and will only be discussed briefly.

The first approach is to allow and encourage learners to speak in their home or main language (de Araujo et al., 2018). Swain, Kirkpatrick and Cummins (2011) observed that in some situations "(i)t is a waste of time to tell students not to use [their first language] when working through cognitively/emotionally complex ideas, as they will do so covertly if not allowed to do so overtly" (p. 15). Setati (2012) calls for learners to be encouraged to use their main language(s), not simply for them to be permitted to use them. One strategy she proposes to do this is to provide them with examples of work in their home language (Setati, 2012). Another strategy, proposed by Prediger et al., (2019), is that the teacher talks in both the language of instruction and the learners' home language. Setati (2012) notes that while the "deliberate, proactive and strategic use of the learners' home languages together with English in the teaching and learning of mathematics in multilingual classrooms" (p. 144) proved successful in the study, more research is required to establish whether this approach can be used more widely.

The second approach is to teach the new language explicitly. Tavares (2015) identified a number of explicit teaching strategies used by a bilingual teacher, teaching mathematics in English, to learners who were still learning English. For example, the use of syllabification (breaking words down into syllables, to help learners with reading and pronunciation), and the use of morphological cues, such as teaching learners the meaning of a prefix such as *tri-*. Another strategy that was identified was the promotion of vocabulary-building through keeping a so called dictionary of new terms with explanations in the learners' home language.

The third approach also involves making language explicit, but in this approach the *differences* between the languages are made explicit. Edmonds-Wathen et al. (2016) argue that teachers need to be aware of mathematical ideas that are expressed differently in the language of instruction and in a learners' home language. Being aware of these differences and making them explicit for learners can reduce confusion (Edmonds-Wathen et al., 2016). Paper 2 (Comparison phrases) and Paper 4 (Compare type word problems) in this study provide some insights into the differences in how isiXhosa and English express comparison, thus equipping English teachers of isiXhosa-speaking learners with some of the knowledge necessary to potentially reduce confusion for isiXhosa learners who are taught mathematics in English.

The second and third approaches involve explicit language teaching. The dilemma of how explicit to make language teaching was mentioned in the section on "monolingual contexts" but is just as pertinent, if not more so, for multilingual contexts. Adler (1999) identified the "dilemma of transparency" in her work with English-speaking teachers who had recently started teaching learners whose main language was not English. In order to support these learners, teachers had started using "explicit mathematics language teaching" (Adler, 1995). While teachers observed that teaching mathematics language explicitly and focusing on instructions and explanations, helped all learners, one teacher in particular felt that the focus on what and how something was said was at the expense of the mathematics under consideration. Adler (1999) gives examples of the kinds of decisions teachers are faced with but finally concludes that "there is no resolution to the dilemma of transparency for mathematics teachers; there is only its management through awareness and careful instructional moves when making talk visible in moments of practice" (p. 63).

The fourth, fifth and sixth approaches do not use explicit language teaching. The fourth approach involves the use of gestures. Research suggests that learners, especially young learners, can benefit from the use of nonverbal communication like gestures – both as a means to express themselves (de Araujo et al., 2018) and as a means for teachers to clarify what they mean (Moschkovich, 1999). The fifth approach involves creating a supportive environment for having discussions about what learners believe and why they believe it. In order to learn a new language or register, children (and adults) require opportunities to practice using the new language (Tavares, 2015).

The final approach, revoicing, has already been referred to in the section on monolingual classrooms. In a monolingual context, teachers typically revoice using the correct mathematics register. In a "multilingual context", teachers revoice both when learners use the mathematics register incorrectly and when they make grammatical or other mistakes in the language they are still learning. For example, Tavares (2015) describes how a teacher revoices 'Sine θ over Cosine θ is equal … is equal 1' as "Sine θ over Cosine θ is equal to 1".

Affordances of learning mathematics in two languages

As mentioned previously, when describing the challenges faced by learners moving from their first language to the language of instruction, it is easy to adopt a deficit model to describe learners (Garcia & Gonzalez, 1995). However, by adopting a situated-sociocultural perspective, Moschkovich (2002) reframes learners' languages as resources, showing how their everyday register and home language can be "used as resources for communicating mathematically" (p. 196).

While a common argument for encouraging learners to use the full extent of their linguistic resources, or "language repertoires", is increased participation, Prediger et al., (2019) note a number of other, more nuanced advantages of intentionally combining and comparing languages. For example, comparing or contrasting languages can help find useful distinctions in mathematical understanding, can enrich understanding of mathematical concepts and switching between languages can connect different conceptualisations (Prediger et al., 2019). These advantages have been observed and applied in a recent study of Turkish-speaking German students who were encouraged to use Turkish to engage with fractions problems, even though they had only been taught in German. The study identified differences in how languages express the notion of a fraction, and showed how bilingual learners have access to these different understandings (Prediger et al., 2019).

Based on previous work, Prediger (2019) calls for research that can be used to prepare teachers for “language-responsive mathematics teaching”. Such research should collect and systematize examples of the potentially rich differences in how different languages express mathematical ideas. While both the learners and the teachers are bilingual in Prediger et al.'s, (2019) study, the value of this kind of research is not limited to such contexts. For example, in South Africa, research for language-responsive mathematics teaching could be applied in higher grades where learners are taught in a language different to their main language and where they are able to compare languages, thus extending the approaches proposed by Setati (2012). But it would also be helpful in the lower grades where, even if learners are not in a position to compare languages, learners would benefit from teachers who are aware of language differences. Paper 2 (Comparison phrases) and Paper 4 (Compare type word problems) in this study contain elements of research that can prepare teachers for language-responsive mathematics teaching, thereby contributing to answering one of Setati's (2012) questions: “What do all teachers need to know, and what skills do they need to develop, in order to be able to teach mathematics effectively in multilingual classrooms?” (p. 145).

3.3.2.3 Multilingual context with teacher not fluent in mathematics register

The previous two sections give a sense of the challenges and affordances of teaching and of learning mathematics in two contexts. The first is monolingual or linguistically homogenous classrooms, where the language of instruction has an established mathematics register (such as English, Spanish, French, Chinese) and where teachers are fluent in both the language and the mathematics register of the language. The second section considered “multilingual contexts” where the learners and/or the teacher had access to more than one language. The teaching strategies described in that section assumed that the teacher was fluent in the language of instruction and in the mathematics register of the language of instruction.

In this section, a specific multilingual context that has received less attention is considered. This is the context where monolingual (or emergent bilingual) learners are taught in their home language by a bilingual teacher who shares their home language, but who is not fluent in the mathematics register of the language of instruction. As Gutiérrez (2002) points out, being bilingual does not mean that one is bilingual in mathematics and having a bilingual translation is not sufficient for helping learners develop a mathematics register in an additional language.

A teacher might be fluent in a language but not fluent in the mathematics register of that language for a number of different reasons. One reason could be that they did not study mathematics in that language. This is the case in the United States of America where some teachers are fluent in Spanish but are not fluent in the mathematics register of Spanish because they did not study mathematics in Spanish (Gutiérrez, 2002).

A second reason could be that the language does not have a fully developed mathematics register, often because it is not an official language of instruction. As pointed out by Schleppegrell (2007), while all languages have the potential to develop registers for teaching mathematics, for historical reasons, not all languages have been developed such registers . For example, a Tswana teacher in South Africa describes how she or he "runs out of words" when trying to explain advanced math in Tswana (Adler, 1998). Similarly, in a study by Essien et al. (2015) teachers indicated that the technical mathematical terms that were being introduced in their classes in the home language of the learners, were terms they themselves had not encountered previously, in spite of being proficient in the home language in question.

A third reason could be that the language in education policy of a country changed so that a teacher with an indigenous language as home language was taught mathematics in a colonial language such as English, but is now required to teach learners in the indigenous language. This is particularly true in previously-colonized countries where there have been attempts to develop indigenous languages to enable school mathematics to be taught by means of the indigenous language (Trinick, 2015). In many African countries such as Kenya, Malawi and South Africa, learners are now taught mathematics in an indigenous African language for their first few years of formal schooling (Essien, 2018).

One important example of mathematical language development is in New Zealand where a Maori mathematics register was developed in order to be able to teach mathematics in Maori up to the end of secondary school (Barton, Fairhall, & Trinick, 1998). Once the register had been developed, teachers had to be trained to use it as they had not been taught school mathematics using that register because it had not been developed at the time when they were in school.

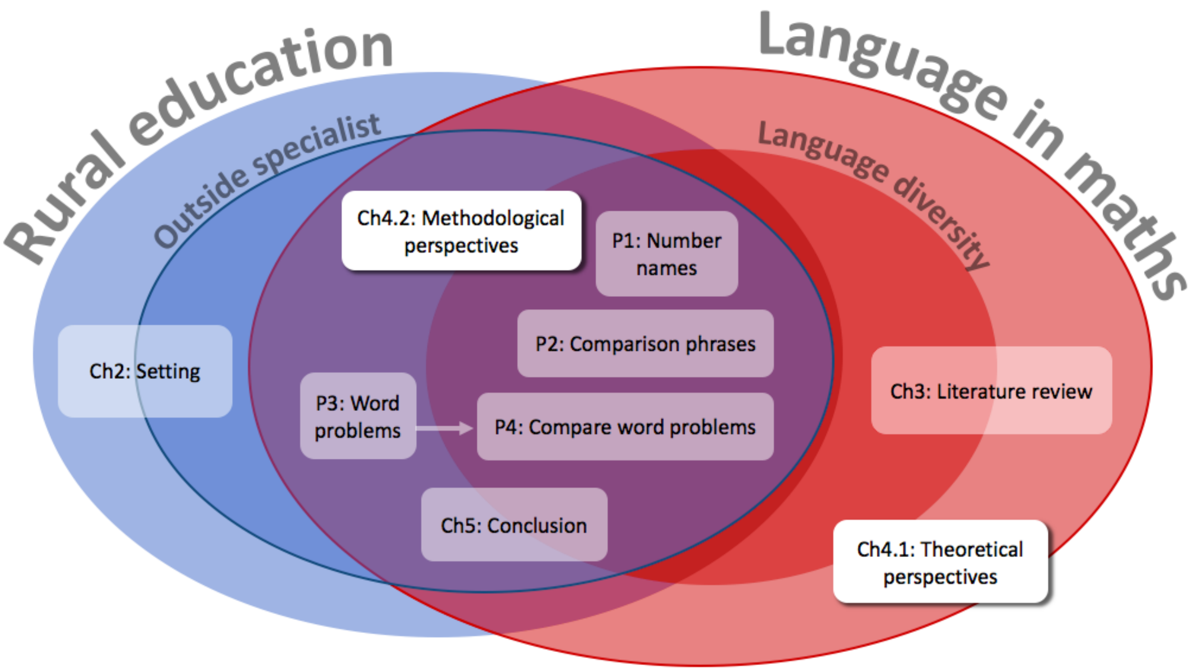
Even in countries where mathematics registers have been developed, either through a rigorous process like in New Zealand or through a more ad hoc process like in South Africa (where the translation of mathematical texts has contributed to developing the register), two problems remain. The first is that tertiary training is rarely offered in an indigenous language. Therefore, teachers are taught how to teach mathematics in a language (e.g. English) different to the one they will use to teach mathematics (e.g. isiXhosa). As mentioned previously, this is slowly changing in South Africa where two universities are now offering bilingual courses (English and an African language) for initial teacher training. The second problem is that even in a case where the mathematics register has been developed or is being developed, for example, in South Africa, this does not yet mean that the intermediary mathematical language is agreed on and widely used.

3.4 Conclusion

This chapter is the second chapter locating the study. The first chapter located the study geographically and socio-politically within rural isiXhosa dominant schools, striving to be accountable to rural learners but limited to the research opportunities available to an "outside specialist". This second chapter has located the research within the field of language in mathematics, focusing particularly on differences in how languages express mathematical ideas. The four papers that form the core of this study fall within the intersection of these two domains.

It is important to note that while much of the current research regarding the relationship between language and mathematics takes as its unit of analysis the *spoken interactions* between learners and between learners and their teacher, the papers in this study take as their unit of analysis *written language*, as it appears in canonical texts. This is primarily due to my limited access to isiXhosa, and means that the study is able to note certain differences between how isiXhosa and English express mathematical concepts such as number and comparison, but is not able to speak to how these differences play out in classrooms. The use of canonical texts as a unit of analysis and the theoretical and methodological perspectives adopted to study them are the focus of Chapter 4.

CHAPTER 4: THEORETICAL & METHODOLOGICAL PERSPECTIVES



At the core of this study are four published papers knitted together by a common theme of the affordances and constraints of teaching mathematics in early grade rural isiXhosa-dominant classrooms, with a specific focus on linguistic resources. In the first two chapters I located this study geographically in rural schools and theoretically in the domain of language diversity in the field of language in mathematics. I also described my position as "outside specialist" and the limitations and affordances this places on the kind of research I am able to undertake.

In this chapter I continue to demonstrate how the four papers are linked, this time by considering the unit of analysis of each paper and the theoretical and methodological perspectives that are common to all four papers. I do this by answering four questions:

- What is the unit of analysis?
- Why is this unit of analysis worth studying?
- How is the unit of analysis being studied, theoretically (with what theoretical lens)?
- How is the unit of analysis being studied, methodologically (with what methodological tools)?

4.1 Unit of analysis

Typically, studies in the field of language in mathematics take as their unit of analysis examples of either spoken or written language. Examples of spoken language are taken from classroom interactions, such as teacher explanations or learner utterances when responding to the teacher or when discussing a solution with peers. These studies are often based on recordings of classroom interactions which are transcribed and analysed according to a certain theoretical framework. For example, in South Africa, Mdluli (2017) transcribed and translated classroom interactions from a Sepedi-medium Grade 3 classroom. Similarly, Prediger et al. (2019) transcribed and translated interactions between Turkish-German speaking Grade 7 learners.

Studies that have focused on written language in mathematics generally fall into two categories, either taking the unit of analysis to be worked examples and problems in textbooks, or to be assessment items in standardised assessments. For example, in their study on textbooks from China and the United States of America, Fuson and Li (2009) compared the learning and teaching tasks used for the same mathematical topic in the two textbooks. However, because the textbooks differed in terms of more than just language (having been written by different writers in two different countries), they did not set out to compare the same mathematical phrases expressed in two different languages.

When studying the influence of language on assessments, the unit of analysis is the items in the assessment. Some studies have demonstrated the effect of changing the phrasing of the items (but keeping the language the same). For example, as discussed previously, Abedi and Lord (2001) demonstrated that rephrasing assessment items to reduce the linguistic complexity of the items improved learner performances on the assessment. Other studies have demonstrated the effect of changing the language of the items. For example, Adetula (1990) showed that Nigerian learners performed better when word problems were administered in their home language than when they were administered in English. For studies such as the one undertaken by Adetula (1990), it is possible to compare the same mathematical phrases expressed in different languages.

In this study, the unit of analysis is examples of written language. Taken as a whole, the unit of analysis across the four papers is "the collection of linguistically expressed mathematical ideas as they appear in early grade canonical mathematics texts in English and isiXhosa". Taken individually, the unit of analysis for each article is a specific subset of all the linguistically expressed mathematical ideas in the canonical texts, in either isiXhosa or English or both languages. Because the canonical texts, including assessments, are available in both isiXhosa and English, it is possible to compare the *same* mathematical phrase expressed in different languages.

The early grade canonical mathematics texts in this study are widely used texts produced by recognized and respected institutions. These include the Curriculum and Policy Statement (CAPS), available in all official languages (Department of Basic Education, 2011b); the Department of Basic Education (DBE) workbooks, also available in all official languages (Department of Basic Education, 2019); the National Education Collaboration Trust (NECT) learner workbooks, available in some of the official languages; and the Early Grade Mathematics Assessment (EGMA) There is one additional set of texts that is included in the study but that do not strictly fit the conditions for being canonical because they are not widely used. These are the early grade mathematics workbooks produced by the Nelson Mandela Institute (NMI). This collection of texts is included in the study primarily as an example of workbooks created not in isiXhosa, but holding the linguistic features of isiXhosa in mind, unlike the other examples which were produced in English and translated into other language versions.

Each of the four papers takes as its unit of analysis the linguistic expressions of a particular mathematical idea as they appear in a number of South African early grade canonical mathematics texts, namely:

- In the first paper, English and isiXhosa number names for whole numbers up to 100, both as used in spoken language and as used in the DBE, the NECT and the NMI workbooks;
- In the second paper, English and isiXhosa comparison phrases in CAPS, and in the DBE, NECT and NMI workbooks;
- In the third paper, additive relation word problems in the English version of the DBE workbooks; and
- In the fourth paper, comparison word problems from CAPS, and from the DBE, NECT and NMI workbooks as well as comparison word problem assessment items from the isiXhosa version of the EGMA assessment.

4.2 Rationale for studying canonical texts

There are two key reasons why the examples of mathematical phrases in canonical texts provide a valuable source of knowledge for the field of language in mathematics. The first reason is the central role canonical texts play within education. The second reason is that, in certain contexts, the texts are available in multiple languages, thus allowing for language-based comparisons of mathematical expressions.

4.2.1 Centrality of canonical texts

In any country, canonical texts such as the curriculum statement are central in defining the intended curriculum (Porter & Smithson, 2001). In South Africa, other canonical texts such as the DBE, NECT or NMI workbooks, which are based on the Curriculum and Assessment Policy Statement (CAPS), influence the enacted curriculum (Porter & Smithson, 2001). This is because the workbooks, rather than the curriculum statement, are often teachers' primary reference regarding what should be taught and how it should be taught. In South Africa, the CAPS is not only used to inform government funded workbooks, such as the DBE and NECT workbooks, but also any other textbooks or workbooks that are produced. In many ways the CAPS and national workbooks exemplify what "doing school mathematics" means in South Africa. Similar canonical texts from other countries exemplify that country's particular version of "doing school mathematics".

While the centrality of the canonical texts is relevant in all four papers in this study, it is particularly relevant in Paper 3 (Distribution of word problems). Paper 3 considers the distribution of word problems in DBE workbooks because, as discussed in Chapter 2, in many rural classrooms the examples in these workbooks are likely to be the only examples of word problems the learners encounter. In doing so Paper 3 builds on the work of Roberts (2016a) who studied the range of word problems in a different canonical text, namely the Curriculum and Assessment Policy Statement (CAPS); and on the work of Galant (2013) and Hoadley and Galant (2016) who took the DBE workbooks as a unit of analysis.

4.2.2 Canonical texts allow for language comparison

Part of the problem behind the problem in South Africa, as described by Ramadiro and Porteus (2017), is that mathematics education specialists in South Africa (including those responsible for curriculum and workbook design and development) are not embedded in the instructional practices and linguistic resources of schools in the mainstream system. The majority of early grade mathematics education scholars are not fluent in an indigenous African language. While there is a small but growing number of mathematics education scholars who do have an African language as their home language, the "knowledge project" as a whole does not yet have the capacity to do justice to the affordances of home language instruction in African languages (Ramadiro & Porteus, 2017).

In this context of linguistically disadvantaged scholars, the examples of mathematical expressions from canonical texts, available in multiple languages, provide a valuable resource allowing for comparisons of how different languages use different linguistic features to express the same mathematical ideas. The collection of examples provides an entry point for mathematics education specialists to work together with language specialists in order to understand and describe how mathematical ideas can be expressed in African languages, and to ascertain whether certain ways of expressing an idea are more useful than others when learning and when teaching mathematics in an African language.

Viewed in this way, even though at an individual level there are few people with access to both the necessary mathematical and linguistic knowledge, the South African education system as a whole, mediated by the collection of canonical texts in multiple languages, does have access to this knowledge – knowledge which could be leveraged to strengthen the mathematics education knowledge project in ways that are beneficial to the majority of South African learners.

4.3 Theoretical framework

The unit of analysis for each paper in this study is drawn from linguistically expressed mathematical ideas as they appear in different languages in South Africa's early grade canonical mathematics texts. While each paper has a different unit of analysis, from number names, to comparison phrases to word problems, in all four papers the analysis of these linguistically expressed mathematical ideas is based on the same underlying theoretical framework. I refer to a framework in the sense of "a coherent set of ideas that together emphasise important aspects of a phenomenon, and that constitute an overt theory that can be put to some practical use" (Mason, 2004, p. 12). The framework used in this study draws on aspects of variation theory, which are extended using ideas from typology, a field of applied linguistics.

In the following sections I begin by describing the main components of variation theory. I then look in detail at the three ideas that form the core of my theoretical framework, namely *dimension of variation*, *range of change* and *example space*, and how these are applied in my study. Finally, I explain how ideas from typology are used to extend the idea of an example space by including language as a dimension that can be varied.

4.3.1 Variation theory

Variation theory is a theory of learning developed by Marton and colleagues (e.g. Marton & Booth, 1997; Marton & Pang, 2006; Marton, 2015). According to variation theory, to understand a phenomenon a learner has to be able to discern certain critical aspects of the phenomenon (Runesson & Kullberg, 2010). While a phenomenon can have multiple aspects, certain aspects are essential to understanding the phenomenon while others are not. In order to discern an aspect of something, a learner needs to experience potential alternatives, that is, they need to experience variation in dimensions that correspond to that aspect, while keeping other aspects constant (Al-Murani et al., 2019). As Marton and Pang (2006) explain "one could not discern the colour of things if there was only one colour" (p. 193). In variation theory, the role of the teacher is to expose learners to different examples of the same phenomenon or object of learning where one aspect varies while the other aspects remain unchanged (Al-Murani, 2006).

Within variation theory there are four significant patterns of variation: contrast, generalization, separation, and fusion (e.g. Marton & Pang, 2006; Marton & Tsui, 2004). Contrast allows a learner to compare an object of learning or a feature of that object with something it is not. For example, in order to understand what "three" is a learner must experience something that is not three, such as "two" or "four". However, to fully understand "three" a learner must learn to generalize by experiencing different variations of "three", for example three apples, and three cars and three houses. If an object of learning has different aspects, separation allows a learner to discern one aspect of an object of learning from other aspects by varying only the aspect of interest while keeping all other features constant (Bussey et al., 2013). For example, in order to understand that volume and mass are two different aspects of density, learners must experience the density of materials with varying volumes but the same mass, as well as experiencing the density of materials with varying masses but the same volume. However, usually both aspects are varying and therefore they need to be "fused" in order for a learner to fully understand density.

In this study, the idea of separation, i.e. discerning the different aspects of the object by varying only the aspect of interest while holding all other aspects constant, is a central construct. However, instead of a teacher providing carefully selected examples with which learners engage, in this study I analyse a set of examples of the object of learning in order to discern the dimensions of variation of the object.

4.3.2 Range of permissible change and example spaces

While variation theory is a general learning theory that can be applied to learning in any field, it is particularly well suited to mathematics education. This is because in mathematics, concepts are usually introduced to learners through examples which have some features in common but which vary across "representations, orientations, numerical content and so on" (Al-Murani et al., 2019, p. 8).

Even before Marton and colleagues (e.g. Marton & Booth, 1997; Marton & Pang, 2006) formulated variation theory, scholars noted that any example has "some attributes that are intended to be exemplified and others that are irrelevant" (Zaslavsky, 2014). These correspond to the essential and non-essential dimensions of variation. It is only by defining which are the essential aspects, and by being able to discern these aspects, that it is possible to define what something is an example of, in other words "examplehood is in the eyes of the beholder" (Zaslavsky, 2014, p. 21).

Watson and Mason (2005), working together with Marton and Runesson, have extended the notion of a *dimension of possible variation* to include the *range of permissible change* (Watson & Mason, 2006). The range of permissible change defines the permissible change along an identified dimension of possible variation. For example, as explained by Al-Murani (2006),

> in the expression x + 3, one of the possible components that can vary is the addend (others include: the letter representing the variable, the operator and the order in which the variable and constant appear in the expression), hence this is a "dimension of variation". The values that the addend can take (i.e. natural numbers, negative numbers, rational numbers and so on...) would constitute the "range of permissible change" of the "dimension of possible variation." (p. 25)

The second extension that Watson and Mason have made to variation theory, as applied to mathematics learning, is the notion of an example space (Watson & Mason, 2005). As pointed out by Askew (2016), "working on examples is the bedrock of most, if not all, mathematics teaching" (p. 80). Watson and Mason take a broad view of what constitutes an example, including not only "worked examples" typically used by mathematics teachers to illustrate a concept but also "exercises" worked on by students to practice specific techniques; as well as “illustrations of concepts and principles, such as a specific equation that illustrates linear equations” and any other “raw material” used for inductive reasoning (Watson & Mason, 2005, p. 3). Taking this broad view, Watson and Mason (2005) explain that “examples are usually not isolated; rather they are perceived as instances or classes of potential examples” (p. 51), thereby contributing to an “example space”.

Watson and Mason (2005) distinguish between two primary types of example spaces. The first are personal example spaces which can either be local example spaces, accessible in the moment and triggered by current tasks, or potential example spaces that are built on an individual’s past experiences. The second are conventional example spaces which consist of examples that are “understood by mathematicians, and as displayed in textbooks, into which the teacher hopes to induct his or her students” (Watson & Mason, 2005, p. 76). Watson and Mason (2005) discuss various ways in which a learner’s personal example space can be expanded, for example, by helping learners discern the different dimensions of possible variation of the example, and exploring the range of permissible change in a particular dimension. In other words, in the language of variation theory, by using “separation”. This idea of expanding an example space by separating the different dimensions of variation and then exploring the range of permissible change in each dimension is a central component of the methodological framework used in this study, as will be discussed later.

4.3.3 Example spaces and canonical texts

In this study I apply Watson and Mason's (2005) notion of an example space to a narrower conception of example, namely to linguistically expressed mathematical ideas as they appear in early grade canonical mathematics texts. In each paper I consider a different mathematical idea and the words (e.g. number names), phrases (comparison phrases), or problems (additive relation word problems) used to express that particular idea linguistically. Therefore, the example spaces in this study do not contain worked examples or exercises but rather words, or phrases, or word problems. Similarly, I use a narrower definition of example space. While Watson and Mason’s (2005) example spaces include start up examples, model examples, counter examples etc. (p. 64), in this study an example space is the set of examples (linguistically expressed mathematical ideas) generated by varying each dimension through that dimension’s range of permissible change.

The conceptualisation of an example space used in this study is closest to what Watson and Mason (2005) refer to as the conventional example space, although their conventional example space is not limited to examples appearing in mathematics texts. As explained previously, Watson and Mason distinguish between a local personal example space as the examples that are accessible to an individual at a given point in time; and a potential example space as the bigger set of examples that an individual is able to access. In a similar way, I distinguish between the *current canonical example space* as found in canonical texts and a *potential canonical example space*. The current canonical example space consists, as the name suggests, of the examples that are currently found in a particular set of canonical texts. However, as is demonstrated in this study, current canonical example spaces do not always include *all* the types of examples that are available in the potential canonical example space.

In the same way that learners can expand their personal example spaces by starting with the current examples they have, identifying the dimensions of possible variation and then expanding the range of change for each dimension; the current example space of canonical texts can be expanded by starting with the existing examples, identifying dimensions of possible variation and considering the range of permissible change for each dimension. In this way a potential canonical example space, or a typology of examples, is set up, as demonstrated in Figure 4.1.

Note that the narrow conceptualisation of an example space used in this study, namely the set of examples generated by varying each dimension through that dimension's range of permissible change, can also be described as a typology. My use of the term "typology" comes from the early literature on additive relation word problems and the attempts to generate and study the full range of different types of additive relation word problems (see for example Carpenter and Moser, 1983; Riley, Greeno and Heller, 1984). For this reason, at some points in this study, particularly in Paper 3 (Distribution of word problems) I use the terms typology and examples space interchangeably. This use of the word typology, namely as an alternative to example space, should not be confused with linguistic typology, a domain of linguistics that I draw on in order to consider language-based differences in how mathematical ideas are expressed, as discussed below.

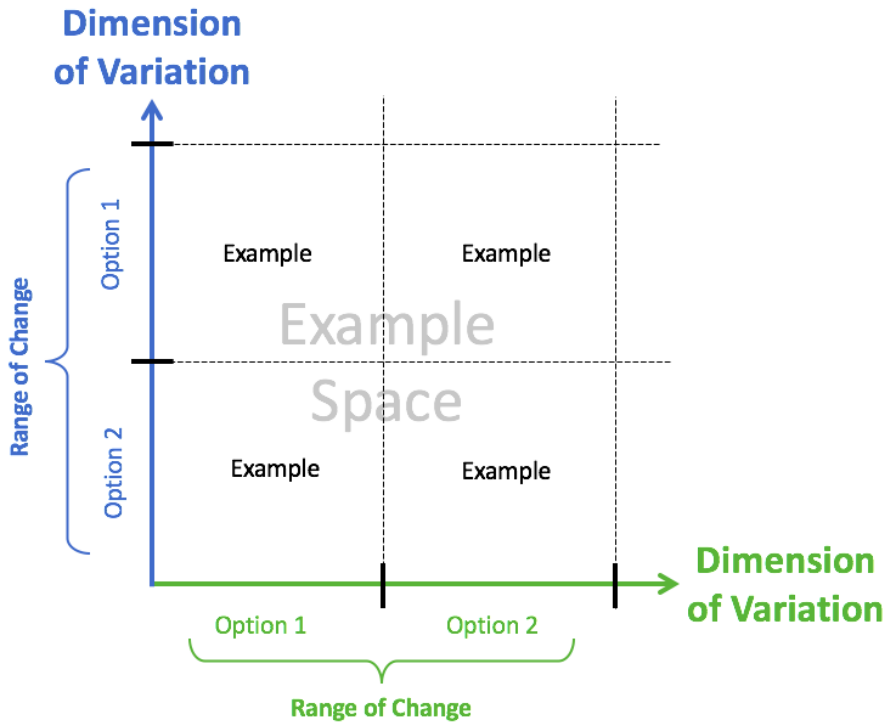


Figure 4.1: Example space generated by changing dimension of variation

Once an example space or typology is set up, it is possible to locate the current examples in the example space and to decide whether the examples that are currently missing from the canonical text should be included. The manner in which I applied this approach in this study is described in more detail in the section on methodological perspectives.

4.3.4 Language as a dimension of variation

In all four papers, I extend variation theory to include examples from canonical texts in example spaces. However, there is a second extension that is used in the three papers that focus particularly on language-based differences. This extension considers the language used as an important dimension of variation when studying how mathematical ideas are expressed. Here I use language to mean "a system of communication used by a particular country or community" rather than "the method of human communication, either spoken or written, consisting of the use of words in a structured and conventional way" (Morgan et al., 2014). By keeping the other dimensions constant (in other words, by looking at the same mathematical phrase expressed in different languages), it is possible to discern the effect that a difference in language has on the way the mathematical idea is expressed.

Variation theory emphasizes that unless a particular dimension is varied, it is difficult to discern that dimension or the influence of that dimension. For example, for linear functions with $y = mx + c$ there are a number of different dimensions of variation such as the value of m and c. If learners only experience examples of linear functions where $c = 1$ they are unlikely to attribute the c as determining the y-intercept of the straight line. However, if c is varied systematically while m is kept constant then learners have an increased chance of seeing the influence of c on the graph of the straight line (Al-Murani, 2006).

Similarly, when the way a mathematical idea is expressed is only considered in one language, it is difficult to see the influence that language has on how mathematical ideas are constructed and expressed. By considering the way in which the same idea is expressed in different languages, it becomes possible to see this influence, and to see the affordances that a particular language has for teaching and learning mathematics. For example, Bartolini Bussi et al. (2014) and Prediger et al. (2019) consider the way different languages express fractions, showing how different expressions emphasise different components of the complex notion of a fraction.

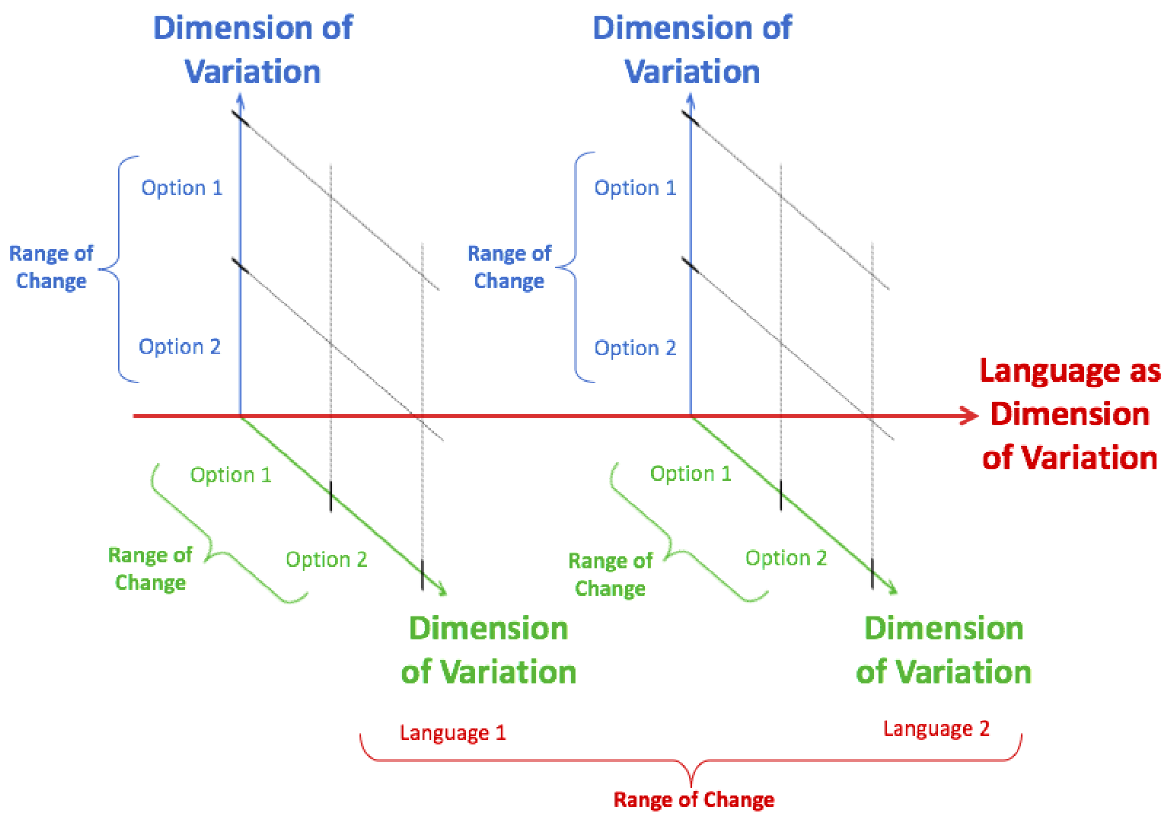


Figure 4.2: Language as dimension of variation

There are two reasons why this extension is particularly apt in this study. The first is that in South Africa, a large corpus of mathematical texts have already been translated into other languages, thus providing a rich source of examples to compare. The second is that as part of the study, examples from these texts are already located within an example space. It is therefore possible to not only vary the dimension of language for one example, but to vary it for a set of examples, as represented in Figure 4.2.

4.3.5 Typological framing

Extending variation theory to include language as a dimension of variation in order to compare examples of mathematical ideas expressed in different languages is informed by ideas from typology, a domain of applied linguistics. Linguistic typology is the systematic study of the similarities and differences of the world's languages (Dixon, 2010). While there are many similarities between languages in terms of structure and form, there are many remarkable ways in which languages differ (Edmonds-Wathen et al., 2016). In as much as typologists study the similarities and differences between languages, typology and variation theory are built on similar ideas. It is only by studying the diversity of languages that linguists are able to understand the full range of what language can be.

A central component of typology is the importance of comparing languages in ways that do not privilege one language over another. This is done through "developing a body of analytically compatible concepts…valid across all the world's languages" (Evans & Dench, 2006, p. 4) and by using a "framework-neutral analysis" (Nichols, 2007, p. 236). As explained in Chapter 3, it is because of this neutrality that Edmonds-Wathen (2019) proposes the adoption of a typological framework to investigate language-based differences in mathematics education. Edmonds-Wathen (2019) argues that adopting such an approach can help with "the negotiation of conceptual or structural inequivalence that are often encountered when mathematics educators try to map directly from mathematical expressions in one language to another" (p. 122). It is this central component of neutrality and the possibility that it brings of "positioning languages with equal status" (Edmonds-Wathen, 2019, p. 122) that is the first idea from typology that informs my extension of variation theory to include language as a dimension of variation.

In order to exemplify a typological framing of mathematics education research, Edmonds-Wathen (2019) gives an example of how a typological approach can be used to study language diversity in mathematics by considering the use of prepositions. Prepositions play an important role in mathematics in English. For example, the spatial meaning that many prepositions have can be used directly in mathematics but can also be used metaphorically to talk about numbers coming before or after other numbers or to say things like "five goes *into* 20 four times" (Edmonds-Wathen, 2019, p. 122). There are, however, a number of languages, such as Pitjantjatjara, that have few prepositions. While this could be seen as a constraint for learning or teaching mathematics in those languages, a typological approach foregrounds the function of the prepositions, not the prepositions themselves. Informed by this approach, Edmonds-Wathen (2019) proposes that because Pitjantjatjara uses the case system to describe the relationship between objects (in the same way that English uses prepositions), the case system can also be used for mathematical functions where English uses prepositions.

Edmonds-Wathen (2019) points out that not all differences between languages can be mapped using a typological approach but that even so, a typological approach provides "a more productive position to investigate the mathematical affordances of a language than when, say, English is the only point of mathematical and linguistic comparison" (p. 123). The focus on the function that a type of word plays in a language, rather than on the form of the type of word, is the second idea that I take from typology, in extending ideas from variation theory to include language as a dimension of variation.

4.4 Methodology

In the previous sections I have discussed the units of analysis used in the papers at the centre of this study as well as the overarching theoretical framework used to study the unit of analysis in each paper. In this section I discuss the two components of the study's methodological approach. The first component constitutes the overarching approach I adopted in order to gain a better understanding of the instructional practices and linguistic resources of rural, isiXhosa-dominant schools. This was an ongoing process which informed the research undertaken in each of the four papers. The second component is the cross-cutting analytical approach I used, to a greater or lesser extent, in each of the four papers. I end this section by discussing the ethical considerations that were taken into account in this study.

4.4.1 Identifying the questions

The problem behind the problem of the two very different education systems in South Africa, as articulated by Ramadiro and Porteus (2017), is that the knowledge held and built by education specialists is not accountable to the socio-cultural and linguistic realities of the majority of South African learners. Most education scholars, myself included, are not embedded within the mainstream education system, having themselves been educated and having taught in English or Afrikaans within the privileged education system. Not being embedded in the realities of the classrooms that they are studying, has a number of disadvantages for education researchers. One of these is that they are not well placed to identify relevant and important issues. Because of this, the first component of my methodological approach relates to how I embedded myself, to a limited extent, within rural isiXhosa-dominant classrooms.

In my role as numeracy coordinator at the rural education non-governmental organisation (NGO), I was fortunate to spend many hours in the Foundation Phase mathematics classes of two rural isiXhosa-dominant schools. Because of the time I spent in the classes I was able to better understand the instructional practices shaped by "overwhelming forces" faced by the teachers (Paxton, 2015, p. 213). I also observed numeracy clubs lead by isiXhosa speaking adults for isiXhosa speaking children. In total I observed over 200 lessons (not all of which involved teaching) and over 100 numeracy club sessions between January 2017 and November 2019.

During this time I not only observed lessons and clubs in which isiXhosa was being used to teach mathematics, I also used two elicitation methods, described by Edmonds-Wathen (2019), to engage with isiXhosa speaking adults about how to express certain mathematical ideas in isiXhosa. The first method was direct questioning. I would ask isiXhosa-English speakers "How do you say 'Three hundred and fifty one' in isiXhosa?" or "How would you say 'Sive has three more apples than Sinovuyo?'". Because so many of the isiXhosa-English speakers I asked struggled to express comparison phrases and questions in isiXhosa, I also used a second method which can be viewed as a "weak form" of the structured elicitation tasks from cognitive linguistics (Edmonds-Wathen, 2019), described in Chapter 2. Rather than asking isiXhosa-English speakers to translate a question, I set up a context "Sive has five apples and Sinovuyo has three apples" and then asked "What question could you ask about this in isiXhosa?". While this method is not as rigorous or structured as the elicitation tasks from cognitive linguistics, it is similar in that a situation is created in which participants are free to use language in a meaningful way.

During my observations of numeracy clubs I was able to benefit from a variation of this second method. Part of my role was to design and share ideas for the weekly numeracy clubs. This was done in English and each week included a compare activity, for example asking the learners to make two lines, one with all the girls and one with all the boys and then asking the learners to work out how many more girls (or boys) were present than boys (or girls). These activities were explained and demonstrated to the numeracy club facilitators in English. The facilitators then explained the activities to the children in their club in isiXhosa. By observing the clubs I was able to gather examples of how comparison phrases and questions were being spontaneously expressed in isiXhosa.

These observations and conversations informed the topics that I chose to study in the four papers. For example, Paper 1 on numbers was informed by my coming to understand how numbers are named in isiXhosa, my observations of the widespread use of English number names and the fact that teachers did not appear to leverage the affordances of the isiXhosa number names (as observed by Mdluli (2017) in relation to Sepedi), together with the early success of leveraging these affordances in numeracy clubs. My observation of the central role played by the DBE workbooks in determining the enacted curriculum, together with my knowledge about the poor performance of learners on word problems nationally, led to Paper 3 on the distribution of word problems in the DBE workbooks. My early, widespread observations that isiXhosa speakers often found it difficult to construct or translate a statement or a question about the difference between two quantities, informed my decision to study comparison phrases in Paper 2 and Paper 4. In particular, observations of the use of the verb *-shota* 'to be short of' by colleagues running numeracy clubs, led to the work in the fourth paper on the impact of different ways of phrasing comparison questions in isiXhosa.

While I can justify the choice of topics and how they are important within the context of rural, isiXhosa-dominant schools, it is important to note that in many ways I am still an outsider, deciding on behalf of teachers within the system what is worth studying. To be truly accountable to the realities of teachers and learners researchers need to work much more closely with teachers to identify issues that they feel are important.

4.4.2 Answering the questions

Before describing the cross-cutting approach I used to answer the central question, it is helpful to note that even though Edmonds-Wathen (2019) describes an implicit assumption within mathematics education that researchers will only study and compare languages that they are fluent in, this is not always the case. In mathematics education, as in linguistics, it is possible to work with unknown or unfamiliar languages. Edmonds-Wathen (2019) explains that

> Linguists often work with languages with which they have only emerging familiarity. They may work purely in an unknown language, or they may work with bilingual speakers using both a source language and a shared output language. Language data is transcribed and then translated and analysed using patterns in the data with reference to the existing body of linguistic description as relevant. In this way, linguists learn to work with languages that they themselves do not speak, preferably learning to speak them as they work, but not necessarily. (p. 128)

Similarly there are mathematics education researchers who work closely with bilingual speakers to study unfamiliar language. For example, the English-speaking researcher Borden (2013) studying Mi'kmaq and the English-speaking researcher Staats (2007) studying Somali. This study is another example of such work.

Because of my limited knowledge of isiXhosa and because of the large corpus of mathematics texts available in isiXhosa and in English, my methodological approach primarily involved analysing the examples in the canonical texts. This was done by drawing on the notion of dimensions of variation and example spaces in order to set up and compare typologies of examples in different languages. I have previously explained how the notion of an example space can be applied to canonical texts. In this section I describe how I applied the notion of an example space in this study.

Once I had identified the mathematical idea I wanted to study and the canonical texts that I wanted to examine, I systematically worked through the texts identifying all examples of how that idea was expressed (either in isiXhosa or in English). I then identified the dimensions of variation and the range of permissible change in the collection of examples. The process of identifying these dimensions and associated ranges was different for each paper and was an iterative process moving between the examples themselves and other knowledge I was able to access. After I had identified the dimensions and the range of change for each dimension, I set up a typology in the form of a table and populated the table with examples from the canonical texts, thus producing a "current canonical example space". Then, working closely with isiXhosa-speakers, I generated additional "missing" examples, thus creating a "potential canonical example space".

The next step is only applicable for the three papers that compared the ways in which isiXhosa and English expressed mathematical ideas (the first, second and fourth papers). In these papers the implications of language were highlighted by varying the dimension of language but keeping all other aspects of the mathematical phrase the same. In order to compare examples and see the similarities and differences between them, I used a simplified interlinear morphemic gloss.

Interlinear morphemic glossing provides a means of presenting the structure of a phrase from the source language in a target language. If only an idiomatic translation is provided, the structure of the source language is often lost (Edmonds-Wathen, 2019). I make use of a four level interlinear morphemic gloss, as described by Edmonds-Wathen (2019) . In this study, the top level gives the source language, isiXhosa, in sentence form. The second level gives the isiXhosa morphemes (the smallest unit of a language that has its own meaning), the third level gives the morphemic gloss in English (the target language), and the final level gives a free translation in English. By comparing the third level of the gloss with the fourth level, it is possible to identify similarities and differences in how isiXhosa and English express mathematical ideas.

The particularities of how the gloss was done is described in Paper 2 (Comparison phrases) and Paper 4 (Compare type word problems). When Paper 1 (Number names) was written, I was not yet familiar with interlinear morphemic glossing and so I used a direct translation approach, translating all the morphemes of the isiXhosa number names but not using the other conventions of an interlinear morphemic gloss.

To summarise, my methodological approach included: (1) identifying examples in canonical texts; (2) identifying the dimensions in which they vary; (3) identifying the range of change for each dimension; (4) setting up a typology with different dimensions and range of change; (5) populating the typology with existing examples; (6) identifying gaps in the typology; and then (7) generating missing examples (working with isiXhosa-speakers to generate examples missing from the isiXhosa typologies). For the cases where I was comparing languages, the final step was: (8) for a particular example, keeping the example the same and only varying the dimension of language.

4.4.3 Ethical Considerations

Finally I offer a note on the ethical considerations that were taken into consideration in the approach that I used, both in terms of the overarching methodology and in terms of the individual papers.

The observations and discussions that formed an important component of the overarching methodology took place within my work for an education NGO. My work for the NGO involved supporting the Foundation Phase teachers in two schools as well as supporting a team of colleagues who ran home-based after-school numeracy clubs. Teachers were informed that because of my lack of experience in Foundation Phase teaching in rural settings, and because I was not able to speak isiXhosa, that my work, especially initially, would consist of observations and discussions with teachers. Teachers and colleagues were also informed that alongside my work for the NGO I was undertaking my PhD which would be embedded in my work for the NGO.

In terms of ethical considerations for the papers, the first three papers in this study are based on data collected from texts and therefore did not require ethical clearance. The only paper that makes use of data collected from learners is Paper 4 (Compare type word problems), concerning the relative difficulty of compare type word problems. Assessing learner outcomes in two schools was part of my original proposal and related ethical clearance (ethical clearance number 2017-060). However, rather than administering a separate assessment about word problems, I decided it was more ethical to add questions to an assessment that was already being administered because this would minimise the disruption of teaching time. This decision was based on the realisation, early in my research, of the extent of the disruption that administering any sort of assessment is likely to have on a rural Foundation Phase class.

For Paper 4 (Compare type word problems), questions were added to an EGMA assessment administered to Foundation Phase learners in five isiXhosa-dominant schools in the rural Eastern Cape. The EGMA was commissioned as part of ongoing work by the Nelson Mandela Institute (NMI) and was administered by the NGO I was working for. The NMI had signed memoranda of understanding with the schools that clarified that data collected from the EGMAs may be used for academic research. The ethical principles of voluntary, informed consent for participation were applied during the collection of the data, as described in more detail in the fourth paper.

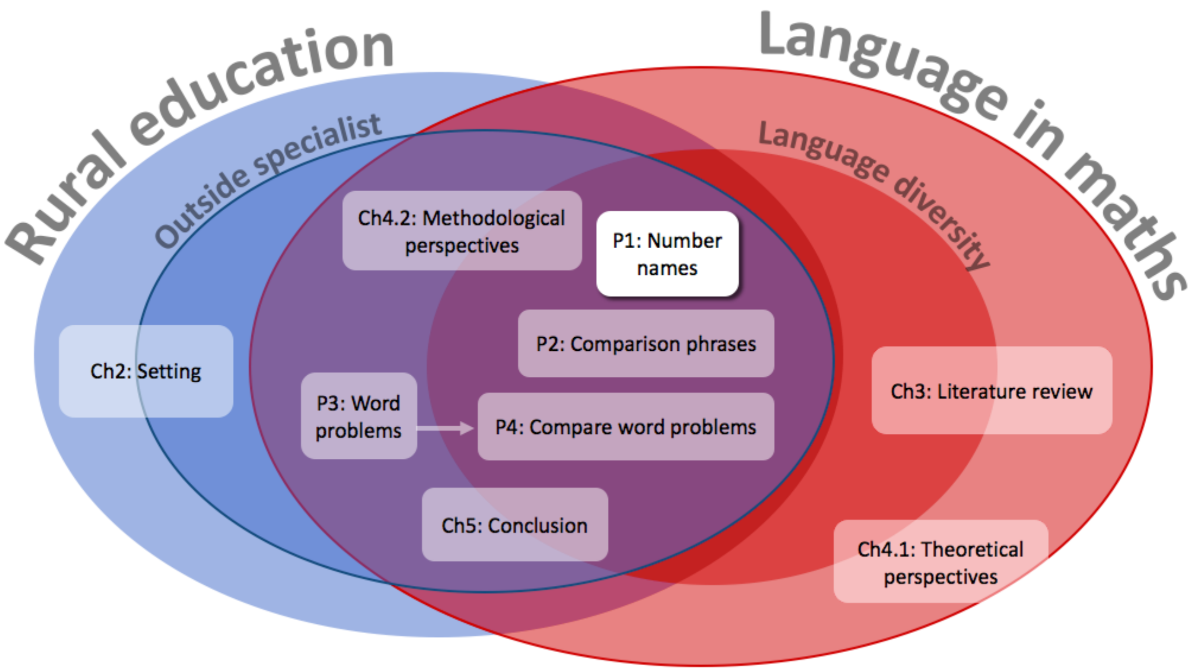
4.5 Conclusion

In this chapter I have discussed the overarching unit of analysis, namely examples of linguistically expressed mathematical ideas found in canonical texts including the CAPS, workbooks and standardised assessments. I argued that written examples of how language is used to express mathematical ideas are an important unit of analysis, even though there has been a shift within mathematics education towards studying spoken discourse. I also demonstrated the value that a large corpus of mathematical text that has been translated into multiple language offers South African early grade mathematics scholars, particularly in terms of identifying the affordances of African languages for the learning and teaching of mathematics.

I presented the underlying theoretical framework which can be seen to have influenced all four papers. This framework draws on ideas from variation theory, particularly the notions of *dimension of variation* and *range of change* as well as the notion of an *example space* generated by considering the full range of permissible change along each dimension of variation. I also extended the idea of an example space to include examples in different languages by framing language as a dimension of variation. To do this I drew on the field of typology from applied linguistics, a field dedicated to comparing languages without privileging one language above another.

Finally, I discussed the cross-cutting methodological approach that I employed. I discussed the overarching approach which led to identifying the research questions in each paper, as well as the cross-cutting approach I used to apply the theoretical framework to the unit of analysis in each paper. In particular, I discussed the two very different additional resources I used when considering language as a dimension of variation - the technique of interlinear morphemic glossing adapted from applied linguistics, and the knowledge of isiXhosa held by isiXhosa English-speaking colleagues, Foundation Phase teachers, numeracy club facilitators, linguists and friends.

PAPER 1: Number names: Do they count?



The first of the four papers that make up the core of this study considers a central component of early grade mathematics learning, namely, number names. This paper is located in the domain of language diversity in mathematics education as it describes the differences between isiXhosa and English number names; as well as the constraints and affordances of each number naming convention for learning and teaching. It also lies within the domain of rural education as undertaken by an outsider, as is evident from the use of canonical texts to study written language. The unit of analysis for the first paper is English and isiXhosa number names for whole numbers up to 100, both as used in classrooms and as used in the DBE, the NECT and the NMI workbooks.

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Number Names: Do They Count?

Ingrid Mostert *

University of Johannesburg, Johannesburg, South Africa
**Email: ingridmostert@gmail.com*

A number of post-colonial countries in Africa have introduced home language instruction for the initial years of schooling. This policy is particularly challenging in mathematics, as there is a paucity of research on how the linguistic features of African languages might be leveraged for the teaching and learning of mathematics. This paper examines the linguistic features of certain African languages, pertaining to number names, by describing five linguistic features of one particular African language, isiXhosa. The five features are: syntactical category, transparency, regularity, length of words and differences between spoken and written language. Different early-grade mathematics texts in English and isiXhosa are compared to explore the implications of these features for the learning and teaching of mathematics. The study concludes that there are both constraints and affordances of learning number names in isiXhosa and other related African languages and suggests how the affordances might be leveraged and the constraints mitigated.

Keywords: *Agglutinative languages; counting; forward number word sequence; linguistic features; number names*

Language and Mathematics Education

The relationship between language and thinking is complex but it is generally accepted that language influences, rather than determines, thought (Whorf, 1956). Within mathematics education, the importance of the relationship between language and mathematical thinking has long been acknowledged (Barwell et al., 2016). The nature of this relationship has been explored by considering how different languages express mathematical concepts. For example, several studies have explored the relationship between language and number, from the early work of Menninger (1969) to the more recent 23rd International Commission on Mathematical Instruction study focusing on whole number arithmetic (e.g. Sun & Bartolini Bussi, 2018). These, and similar studies, are important because the way in which whole numbers are articulated and written in a particular language needs to be considered when teaching early-grade mathematics in that language (Arzarello et al., 2018).

Much of the research on the implications of number naming for mathematics learning and teaching has focused on Indo-European languages such as English (e.g. Mark & Dowker, 2015) and French (e.g. Edmonds-Wathen, Trinick & Durand-Guerrier, 2016) or on the comparison between Indo-European languages and the more regular and transparent East Asian languages such as Japanese and Korean (e.g. Miura, Okamoto, Kim, Steere, & Fayol, 1993) or Chinese (e.g. Aunio, Aubrey, Godfrey, Pan, & Liu, 2008). In comparison, little similar research has been done on African[1] languages. Such research is particularly important in the many post-colonial countries, such as South Africa, where home language instruction has been implemented, at least in the early grades, but where the home languages do not have a long history of being used to teach mathematics (Kazima, 2008).

In South Africa the research on the relationship between language and mathematical thinking has focused primarily on urban multilingual contexts (Adler, 2001; Setati, Chitera & Essien, 2009). Much less research has focused on rural contexts where all learners speak the same African home language and where this language is the language of learning and teaching in the first four years of school (Grade R–3).

In the field of early-grade language education in South Africa there has been a growing acknowledgement of the need for research to support the teaching of reading and writing in African languages, rather than simply translating and adopting materials and approaches from English (Ramadiro & Porteus, 2017). This is not yet the case in early-grade mathematics education, where there is a paucity of research about the learning of mathematics in any language besides English. This paucity of research reflects what Barwell (2003) termed 'linguistic discrimination' in international mathematics education research, which favours 'world' languages such as English and French. Some exceptions include the work of Poo (2017) on early number learning in Sepedi and the work of Chauma (2012) on teaching primary mathematical concepts in Chitumbuka.

This paper sets out to add to the body of research on learning and teaching early-grade mathematics in African languages by focusing on one language, isiXhosa, and one aspect of early-grade mathematics, the naming of numbers. IsiXhosa was selected as the author works in the rural Eastern Cape where the language of learning and teaching in the first four years of formal schooling is isiXhosa.

The implications of the linguistic features of isiXhosa for the articulation of numbers will be considered, as well as the implications for learning and teaching numbers. In doing so, this paper speaks to two of the hindrances to implementing home language instruction in early grades in the South African context. The first is the production of materials. Most early-grade mathematics materials are still conceptualised and written in English and then translated into African languages without much adaptation. This results in the differences between the linguistic features of English and African languages being neither acknowledged nor leveraged. The second is that many teachers who teach mathematics in an African language received their initial teacher education in English and not in the language they teach mathematics in. In order to fully realise the potential benefits of early-grade home language instruction in African languages, these two hindrances need to be addressed.

Research Questions and Methodology

The two questions addressed in this paper are:

(1) What are the linguistic features of isiXhosa that affect the articulation of numbers in isiXhosa, and how do they affect them?
(2) What affordances and constraints do these linguistic features provide for the learning and teaching of numbers in early-grade mathematics?

In order to address the first question, an analysis of the isiXhosa language (in particular of number words) was undertaken. The linguistic features pertaining to number that have been identified in other languages (Edmonds-Wathen et al., 2016) were used as the starting point for the analysis. In order to answer the second question the author drew on data collected during two years of classroom observations in two schools in the rural Eastern Cape. The author observed seven early-grade teachers, all first-language isiXhosa speakers teaching mathematics to first-language isiXhosa speakers in isiXhosa.

While the initial observations related primarily to implications for spoken language, they also pointed to implications for written language as exemplified in the teaching texts used in the two schools. In 2017, the primary teaching text used by both schools was the isiXhosa learner workbooks supplied by the Department of Basic Education (DBE). In 2018, both the DBE workbooks and the teacher toolkit developed by the National Education Collaboration Trust (NECT) were used. One of the schools also used the learner workbooks developed by the Nelson Mandela Institute (NMI) as a source of worksheets. In order to understand the implications of the linguistic features of isiXhosa on written number names, these three isiXhosa texts were compared with each other. To examine some of the differences between the linguistic features of English and isiXhosa with relation to numbers, the English and isiXhosa versions of the texts were compared (all texts were originally written in English).

Linguistic Features of isiXhosa

isiXhosa is one of four Nguni languages spoken in South Africa, the others being isiZulu, isiNdebele and siSwati, and is spoken by more than 8 million South Africans (of a total of 57 million). Like other Nguni languages it is an agglutinative language with a conjunctive orthography (Pretorius, 2018). Like other Bantu languages, it has a noun class system and a resulting system of concordial agreement.

In agglutinative languages with conjunctive orthographies, different morphemes (the smallest unit of a language that has its own meaning) are 'glued' together to form words or sentences. For example, in isiXhosa, the sentence '*Siyabathanda*' (We like them) is made up of the morphemes *si* + *ya*[2] + *ba* + *thanda* (we + them + like).

In isiXhosa, like in other Bantu languages, all nouns belong to a particular class. Nouns are the dominant part of speech and consist of a noun prefix and a noun stem. The class of a noun is determined by the noun's prefix so that all nouns with the same prefix belong to the same class. The system of concordial agreement refers to the manner in which any word (be it a verb, noun, pronoun or adjective) associated with a noun has to show concordance (or agreement) with that noun (Bryant, 2007). This is achieved through adding a concord (a prefix) to the verb, noun, pronoun, adjective, etc., which contains similar-sounding letters to the noun prefix. For example (Bryant, 2007):

***Aba**ntwana **ba**kaJohn **ba**yangxola*	(John's **children** are making a noise)
***Izi**nja **zi**kaJohn **zi**yangxola*	(John's **dogs** are making a noise)

In the first example, the noun is ***aba**ntwana* (children) with the prefix ***aba-*** while in the second sentence the noun is ***izi**nja* (dogs) with the prefix ***izi-**.* The class that a noun belongs to determines all the concords that are used in relation to that noun.

Linguistic Features of isiXhosa Pertaining to Number

Edmonds-Wathen et al. (2016) identified and exemplified linguistic features that can potentially impact on mathematical thinking in relation to number, logic and space and geometry. The linguistic features pertaining to number include:

(1) syntactic category of number;
(2) transparency of number system; and
(3) regularity of number system.

In this section linguistic features of isiXhosa, pertaining to number, are described in detail and the implications of these features for counting and particularly for the forward number word sequence (Wright, 2013) are considered. Each linguistic feature is compared with the corresponding linguistic feature in English.

In the analysis of isiXhosa number names, two additional features are relevant and are offered as an extension to the above framework:

(4) length of number names; and
(5) difference between spoken and written language.

Syntactic Category

Numbers operate in different ways in different languages (Barton, 2008; Edmonds-Wathen et al., 2016). They can operate as nouns (English), as adjectives (Kankana-ey), as nominalised verb phrases that act modally (Yoruba), similarly to verbs (Māori and Mi'kmaq) or as a combination of these (Edmonds-Wathen et al., 2016).

In English, numbers function as adjectives in everyday discourse and as nouns in mathematical discourse (even though they form their own grammatical class) (Barton et al., 1998). In English the same word is used when a number is used as an adjective (e.g. four dogs) and as a noun (e.g. four less than seven is three). Because of this linguistic feature of English, it is not possible to distinguish whether the numbers in the forward number word sequence (FNWS) are being used as adjectives or as nouns. This is not the case in isiXhosa.

In isiXhosa, as in English, numbers can function as numeral nouns and as adjectives.[3] However, in isiXhosa, unlike in English, different forms of the number names are used when numbers function as nouns and when they function as adjectives.

Numbers as Nouns

isiXhosa has numeral nouns for all counting numbers (all whole numbers excluding zero) (see Table 1). Numeral nouns, like all nouns, belong to a particular class and can be singular or plural, for example ***isi****thathu* (three) or ***izi****thathu* (threes). The first eight counting numbers belong to class 7/8 while the numbers nine and ten are in class 5/6. Although isiXhosa has numeral nouns, they are not often used in everyday discourse (Oosthuysen, 2016) nor are they used when counting (both in verbal and resultative counting). Instead, when saying the FNWS, the adjectival form of numbers is used.

Numbers as Adjectives

In order to understand how *numbers* are used as adjectives in isiXhosa it is necessary to give a brief description of how adjectives are used *in general* in isiXhosa.[4]

In isiXhosa an adjective must show agreement with the noun that it is describing. This is achieved by adding a concord to the adjective, one that is similar to the prefix of the noun. Because nouns have different prefixes, different concords are used and so one adjective has different forms. For example, the adjectival stem -*dala* (old) can be used to describe the noun ***ii****kati* (cats) which is in class 10 and it can be used to describe the noun ***ama****gqabi* (leaves) which is in class 6. However, in each case a different adjectival concord is used:

ii*kati*	***ezin****dala*	(the cats that are old, i.e. old cats)
ama*gqabi*	***ama****dala*	(the leaves that are old, i.e. old leaves)

Similarly, the adjectival stem -ne (four) can be used to describe the noun ***ii****kati* (cats) and the noun ***ama****gqabi* (leaves):

ii*kati*	***ezi****ne*	(the cats that are four, i.e. four cats)
ama*gqabi*	***ama****ne*	(the leaves that are four, i.e. four leaves)

Table 1. The use of numbers in different isiXhosa workbooks

	Numeral nouns	Noun class	Adjectival stem or relative	FNWS	DBE workbooks	NMI workbooks	NECT workbooks
1	isinye	7/8[1]	-nye	inye	inye	nye	nye
2	isibini	7/8	-bini	(zi)mbini	zimbini	mbini	mbini
3	isithathu	7/8	-thathu	(zi)ntathu	zintathu	ntathu	ntathu
4	isine	7/8	-ne	(zi)ne	zine	ne	ne
5	isihlanu	7/8	-hlanu	(zi)ntlanu	zintlanu	ntlanu	ntlanu
6	isithandathu	7/8	-thandathu	(zi)ntandathu	zintandathu	ntandathu	zintandathu
7	isixhenxe	7/8	sixhenxe	(zi)sixhenxe	zisixhenxe	sixhenxe	zisixhenxe
8	isibhozo	7/8	sibhozo	(zi)sibhozo	zisibhozo	sibhozo	zisibhozo
9	ithoba	5/6	lithoba	(zi)lithoba	zilithoba	lithoba	zilithoba
10	ishumi	5/6	lishumi	(zi)lishumi	zilishumi	ishumi	zilishumi

[1] Most classes in isiXhosa occur in a singular/plural pair so that the nouns in class 8 are the plurals of the nouns in class 7 and, unlike English, it is the prefix of a noun that indicates whether it is singular or plural.

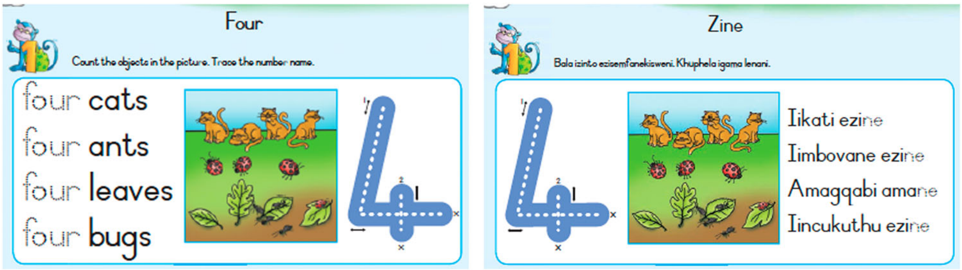

Figure 1. A comparison of 'four' used as an adjective in the English and isiXhosa DBE workbooks in Grade 1 Term 1.

As can be seen in the two extracts from the Grade 1 DBE workbook (see Figure 1), the need for agreement between adjectives and nouns means that, unlike the English where the same word is used throughout the example, in isiXhosa three different forms of the word are used: *zine*, *ezine* and *amane*.

In isiXhosa, as in English, adjectives can be used attributively (the old cats) or predicatively (the cats are old). In each case a slightly different concord is used:

Attributively: ***iikati ezine*** (the cats that are four, i.e. four cats)
Predicatively: ***iikati zine*** (the cats are four)

With a basic understanding of the use of adjectives in isiXhosa it is possible to discuss the words used for the forward number word sequence in isiXhosa.

Counting Numbers

The FNWS (see Table 1) is not grammatically straightforward. Adjectives require a concord and concords depend on the collection of nouns being counted. However, when reciting the FNWS (verbal counting) no specific collection of nouns is being counted. In isiXhosa, what is known as the 'basic noun' from class 9, *-nto* (thing), and its plural from class 10, *-izinto* (things), are used as a general reference to a noun or nouns. So when counting unspecified 'things', the concord for class 9/10 is used (namely zi-). However, the noun 'thing/s' is only implied and not stated and, for various reasons beyond the scope of this paper, the concord is often dropped in the FNWS. This can be seen Table 1 where the DBE workbooks use the prefix zi- when naming numbers, the NMI workbooks drop the zi-, and the NECT workbooks only use the zi- for the numbers 6 to 10.

In isiXhosa, the FNWS uses numbers *predicatively*, therefore the correct direct translation for the FNWS in isiXhosa is: it (the thing) is one; they (the things) are two; they (the things) are three, etc. (see Table 1).

Transparency

The transparency or explicitness of a number naming system refers to the extent to which spoken numbers explicitly state the units that are implied in the written place-value system. These units (ones, tens, hundreds, etc.) are referred to as numeration units (Houdement & Tempier, 2015).

The English number naming system is not entirely transparent for numbers 11–99 as the numeration units for 'ones' and 'tens' are not stated explicitly. For example, from the name 'thirty-seven' it is not possible to infer that there are 3 tens and 7 ones. East Asian languages such as Chinese, Japanese and Korean are completely transparent as all the numeration units are stated explicitly: fifty-six is spoken (and written) as five-tens-six-ones (Sun & Bartolini Bussi, 2018).

A number of researchers have suggested that this transparency is one of the reasons why many East Asian students are more at ease with place value than English speaking learners (e.g. Fuson & Li, 2009; Miura et al., 1993). However, it is possible that the differences are due to other factors such as cultural values, school curricula, classroom instruction, etc. (Sun & Bartolini Bussi, 2018). Studies that have excluded these factors by testing learners from the same country and education system who are taught in different languages (one using transparent number names and the other not, e.g. English and Welsh in Wales or English and Chinese in Hong Kong), have concluded that the transparency of number names does influence certain aspects of early-grade mathematics (Dowker, Bala, & Lloyd, 2008; Mark & Dowker, 2015). Similar studies have not yet been done for other less-known languages that are also transparent, such as Maori (Young-Loveridge & Bicknell, 2015), or African languages such as isiXhosa.

Most African languages fall between Chinese and English in terms of transparency: numeration units are implicit for ones (as in English) but are explicit for tens (as in Chinese) (Sun & Bartolini Bussi, 2018). isiXhosa is no exception.

In isiXhosa, when naming numbers bigger than 10, first the tens are qualified (i.e. the number of tens is given) and then the units are added using the connective concord ***na-*** (ana + isixhenxe = anesixhenxe) which can be translated as 'with'. For example, thirty-seven is written as:

ama*shumi*	***ama****thathu*	***ane****sixhenxe*
tens	that are three	that are with seven

In this example the three (*-thathu*) is being used adjectivally to qualify the numerosity of the tens (***ama-****shumi*), a plural noun in class 6.

The close link between isiXhosa number names (spoken numbers) and the written place-value system (written numbers) could be leveraged to help learners understand the conventions of the place-value system. However, observations of isiXhosa teachers confirm the observations of Sepedi teachers made by, namely that teachers do not generally make use of this linguistic feature of African languages. One way in which this feature could be leveraged, as is done in Chinese mathematics education, is to use the idea that adding two things of the same unit gives an answer of the same unit, e.g. five dogs and two dogs are the same as seven dogs. In the same way five tens and two tens are the same as seven tens (Sun & Bartolini Bussi, 2018). In English this kind of leveraging is not possible for tens (as, say, two tens is spoken as 'twenty') but is possible for hundreds (e.g. five hundreds and two hundreds is the same as seven hundreds).

Finally it is also important to note that the transparency of isiXhosa only addresses the 'positional principle' of the place-value system (Houdement & Tempier, 2015, p. 99) and does not make the multiplicative relationship between numeration units explicit (i.e. that one hundred equals ten tens). This is compounded by a greater focus in the South African foundation phase curriculum and assessment policy statement (CAPS) (Department of Basic Education, 2011) on the positional principle of place value. This focus is shared with many curricula in the West that list place value as positional knowledge only (Sun & Bartolini Bussi, 2018). The problem with this approach to place value is that learners incorrectly answer questions such as 'in 764 ones there are … tens' with 6 instead of 76.

Regularity

A number naming system is regular to the extent that the same 'rules' are used to name all numbers bigger than 10 (for a base-10 number system). Transparent number naming systems such as those discussed in the previous section (Chinese, Japanese, Korean) are typically also completely regular (which might explain why the terms transparent and regular are often used interchangeably). It is, however, possible for an opaque number naming system to be (partly) regular. For example, number names in French for the numbers 80–99 are not transparent (the number of tens is not stated explicitly) but they are regular (the same rules are used to determine the names of numbers). English, which is partly opaque (see previous section), also contains a number of irregularities due to influences of Old Saxon on the names eleven and twelve, as well as due to phonemic modifications as listed by Mark and Dowker (2015):

> In 13–19, ten becomes -teen, three becomes thir-, and five becomes fif-. Above 19, ten becomes -ty for tens starting from 20, two becomes twen- in the twenties and four becomes for- in the forties. (p. 101)

An analysis of isiXhosa number names in Table 2 reveals a number of irregularities such as the difference between five (*zin**tlanu***) and fifteen (*ishumi elinesi**hlanu***) and between fourteen (***i**shumi **eli**nesine*) and twenty-four (***ama**shumi amabini **a**nesine*). However, a closer look at isiXhosa grammar rules reveals that all these apparent irregularities are instances of changes that occur as a result of the system of concordial agreement or as a result of regular phonetic mutations (when two phonemes, occurring next to each other because of agglutination, influence each other).

This difference in the form of the number word used (and consequently the difference in spelling of related number names) is not emphasised in Grade 2 and 3 texts, even though CAPS states that learners should write number names for 1 to 100 in Grade 2 and 1 to 1000 in Grade 3 (Department of Basic Education, 2011). Further research is required to establish to what extent these different forms (and spellings) impact on learner performance and whether there are significant differences across languages when requiring young learners to write and correctly spell number names.

In the following two sections, two additional features of isiXhosa are discussed, namely the length of number words and the differences between spoken and written language. Although the length of number words could be considered as a subcategory of the transparency of isiXhosa number names, because of the important implications for learning and teaching, it is discussed as a separate category.

Length of Words

While the transparency of isiXhosa has potential advantages for supporting learners in understanding the base-10 number system, it also has the disadvantage of longer number names. The longer names are due to three factors: longer names for numbers 1–10; the tens units and hundreds units being explicitly stated for numbers; and the use of concords for numbers bigger than 10.

In English all number names up to ten, excluding seven, only have one syllable. In isiXhosa, the majority of number names up to ten have three syllables. This is compounded for two-digit numbers where, for example, *amashumi amathandathu anesithandathu* (sixty-six) has 34 letters and 15 syllables as opposed to the eight letters and three syllables in English (see Figure 2 for comparison of other numbers).

Longer words impact on learning and teaching as they increase the difficulty of both spoken and written language. This is because the component of working memory that holds information (both spoken and written) in speech-based form (known as the phonological loop) is only able to retain information for 1–2 seconds (Baddeley & Hitch, 1974). This is problematic because it takes more than 2 seconds to say number words in isiXhosa for numbers bigger than 20. Noting the significantly higher demand this places on isiXhosa learners than on English learners raises the question as to whether learners should be tested on the ability to write number names from Grade 1, especially in national tests such as the annual national assessments.

In terms of spoken language, longer number names also increase the difficulty of noticing what stays the same and what changes when counting in tens or in hundreds (Mason & Johnston-Wilder, 2004).

Table 2. (Ir)regularity of isiXhosa number names

1	*inye*	11	*ishumi elinanye*	21	*amashumi amabini ananye*
2	*(zi)mbini*	12	*ishumi elinesibini*	22	*amashumi amabini anesibini*
3	*(zi)ntathu*	13	*ishumi elinesithathu*	23	*amashumi amabini anesithathu*
4	*(zi)ne*	14	*ishumi elinesine*	24	*amashumi amabini anesine*
5	*(zi)n**tlanu***	15	*ishumi elinesi**hlanu***	25	*amashumi amabini anesi**hlanu***

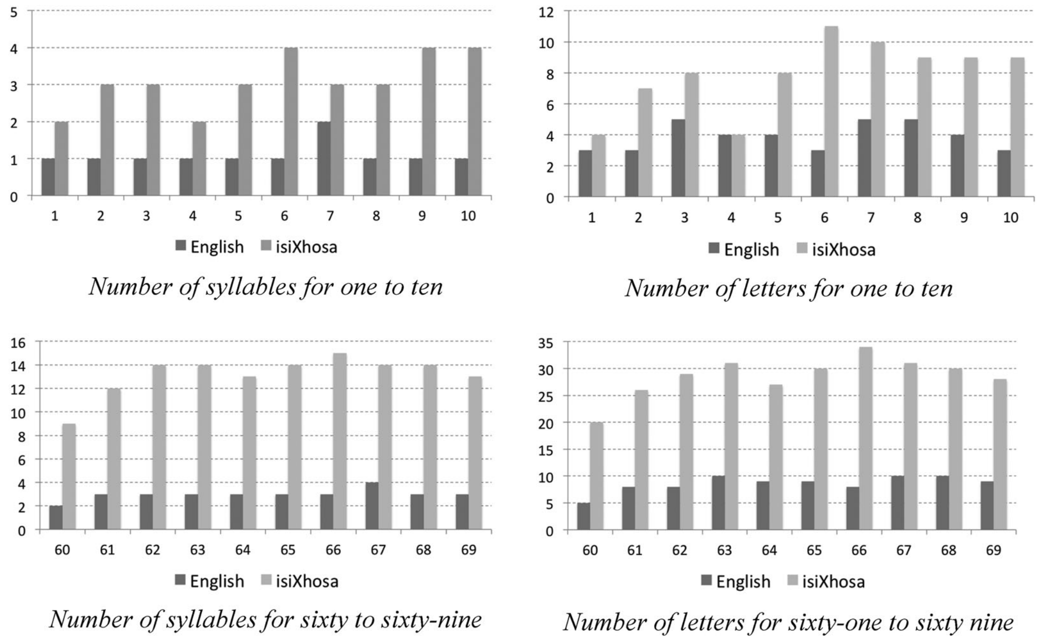

Figure 2. Comparison of English and isiXhosa number words in terms of syllables and letters.

Noticing such patterns can help learners understand the conventions of the place-value system and can help them with mental arithmetic.

Difference Between Spoken and Written Language

The second additional feature of isiXhosa is the difference between spoken and written language. This feature was introduced because of the prevalence of chorusing in the classes observed (Hoadley, 2012) and the prevalence of using English number names in everyday discourse (Oosthuysen, 2016).

In many of the early-grade isiXhosa classes it is a common practice for learners to chorus the numbers from 1 to 100 in isiXhosa and in English, even in the first term of Grade 1. Because different number words have different number of syllables, syllables need to be dropped in order to fit into the rhythm of the chorus. The ‘rules’ determining how the syllables are dropped relate to the common differences between spoken and written isiXhosa. For example, in spoken isiXhosa, when a word ends in a vowel and the next word starts with a vowel, the vowel of the first word is dropped. This results in learners chorusing number words that are phonetically different to the written version of the number words. This difference is significant because, as described previously, learners are expected to correctly spell number words. Further research is required to establish whether learners are likely to make spelling mistakes that correspond to the differences between the chorused words and the written words.

The second difference between spoken and written language is the practice adopted by most isiXhosa speakers of using English number names in everyday discourse (Oosthuysen, 2016). As described by Feza (2016) in a small-scale study, one of the implications of this is that some isiXhosa learners enter grade R knowing the FNWS in English and in isiXhosa or, in some cases, only in English. Another implication is that this practice is adopted in schools so that English number names are used in schools where the language of learning and teaching is isiXhosa. There is currently no data quantifying the extent of this practice but anecdotal evidence suggests that it is widespread.

Discussion

In this paper the linguistic features pertaining to number of isiXhosa, as an example of an African language, were discussed. Firstly, in terms of the syntactic category of numbers, isiXhosa numbers can be nouns or adjectives. However, unlike English, different forms of the number word are used for the two syntactic categories and different forms are used depending on the set of nouns being counted. Secondly, isiXhosa is more transparent than English (as the number of tens are explicitly stated) but less transparent than Chinese (as the unit for ones is not explicitly stated). Thirdly, in terms of the regularity of number names, isiXhosa has some apparent irregularities but these can all be explained by the grammatical features of isiXhosa such as the system of concordial agreement. In terms of the fourth linguistic feature, the length of number names, isiXhosa typically has much longer names (in terms of number of syllables and number of letters) than English. Finally, there are differences between spoken and written number names both in terms of choruses used in schools and in terms of the practice of using English number names in everyday discourse.

These findings suggest that, although there are affordances of the isiXhosa number naming system, there are also considerable constraints for isiXhosa children, compared with English children, when learning the FNWS and when being assessed on number words, in writing. The primary affordance is the transparency of the number naming system for numbers bigger than 10. There is little evidence, however, that teachers are equipped to leverage this transparency to support learners in understanding and using the base-10 number system. The primary constraint is the length of number words (partly as a result of the transparency), which increases the difficulty of identifying patterns in both spoken and written numbers, for example, when counting in tens. The length of the number words together with the changes in spelling increases the difficulty of assessment questions in which learners are expected to read and write number names.

Conclusion

This paper concludes with a number of recommendations regarding how the affordances of the linguistic features of isiXhosa and other related African languages might be leveraged, and how the constraints might be mitigated. First, the practice of using English number words interchangeably with African language number words should be integrated into the teaching of mathematics, rather than discouraged. In this way the affordances of both English and African language number words can be leveraged. Second, teachers should be sensitised to the potentially powerful grammatical features of African languages – particularly that the number naming systems make the number of tens explicit for numbers between 11 and 99, and how this is extended to the number of hundreds for 101 to 999. The use of English number names in schools where English is not the LoLT has very recently been acknowledged and accepted by the DBE in the revision of foundation phase CAPS Section 4 (Assessment) (Department of Basic Education, 2018, p. 6). This is a promising first step but without the implementation of the second recommendation, it runs the risk of encouraging the replacement of African language number words by English number words.

Third, with regard to chorusing, teachers should be made aware of the importance of clearly pronouncing all of the syllables in the number words and ensuring that learners practice pronouncing all of the syllables rather than simply chorusing numbers in a rigid rhythm. Fourth, in terms of assessment, for standardised national assessments, questions which differ considerably in difficulty as a result of difference in length of words need to be carefully considered as to whether they are necessary and fair questions. Finally, the translation of teaching and learning texts needs to be sensitive to the linguistic differences between the language they were written in (English) and the languages they are translated into.

Without the necessary investment into the research required to inform the creation and translation of early-grade mathematics learning and teaching texts, the potential of African languages to support the learning of early-grade mathematics will continue to be undermined.

Disclosure Statement

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Notes

1. In this paper African languages refer to Indigenous African languages.
2. -ya- in this case has no meaning but is inserted because the long form of the positive present tense is used.
3. See *The Grammar of isiXhosa* Chapter 28 (Oosthuysen, 2016) for a more detailed discussion of isiXhosa numbers.
4. Because this is not a linguistics paper, this description refers to attributive and predicative adjectives, which are a Eurocentric oversimplification of how adjectives and relatives are constructed in isiXhosa. See The Grammar of isiXhosa Chapter 8 (Oosthuysen, 2016) for a more detailed discussion of isiXhosa qualificative nouns (adjectives).

ORCID

Ingrid Mostert http://orcid.org/0000-0001-9756-6622

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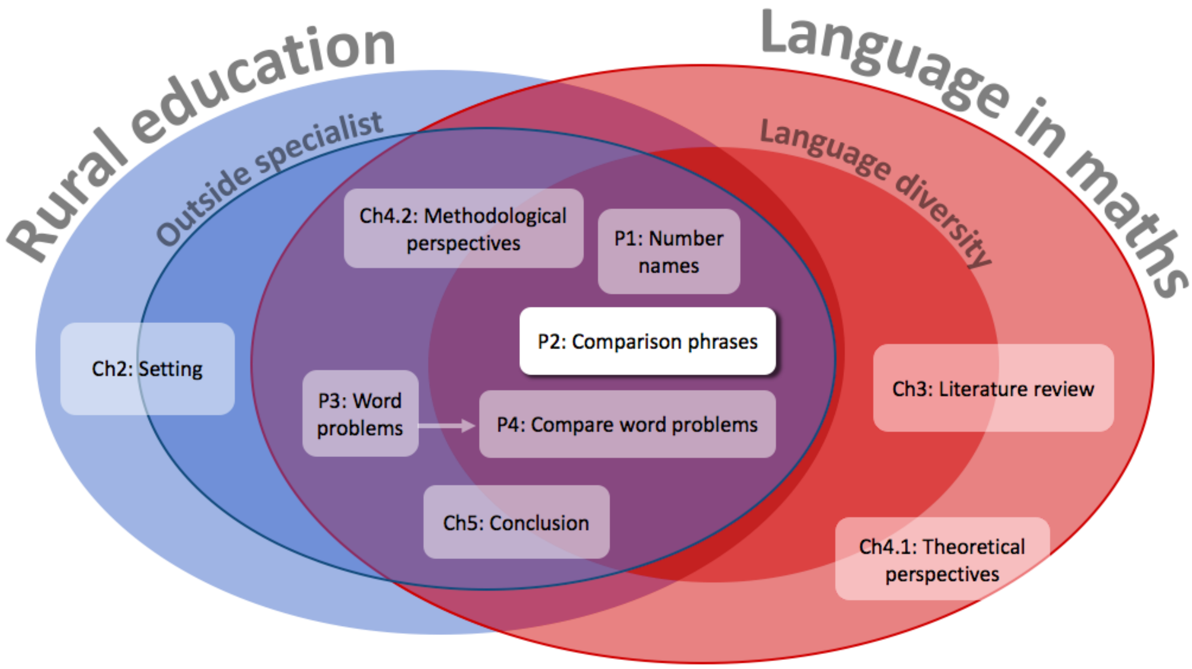
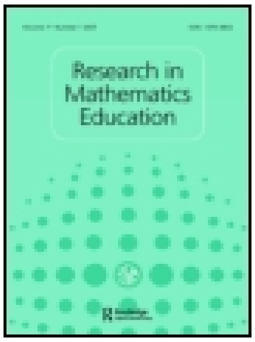
PAPER 2: Diversity of mathematical expression: The language of comparison in English and isiXhosa early grade mathematics texts



The second of the four papers at the core of this study considers another central component of early grade mathematics, namely, the notion of comparison. Like the first paper, it is located in the domain of language diversity, as it uses a typological perspective to contrast the language of comparison of isiXhosa and English, specifically when comparing countable quantities and numbers. It also lies within the domain of rural education as undertaken by an outsider, as is evident from the use of canonical texts to study written language. The unit of analysis of the second paper is the collection of isiXhosa and English comparison phrases in the CAPS, and in the DBE, NECT and NMI workbooks.



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Diversity of mathematical expression: the language of comparison in English and isiXhosa early grade mathematics texts

Ingrid Mostert & Nicky Roberts

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Diversity of mathematical expression: the language of comparison in English and isiXhosa early grade mathematics texts

Ingrid Mostert and Nicky Roberts

Department of Childhood Education, University of Johannesburg, Soweto, South Africa

ABSTRACT
The comparison of different languages in terms of specific mathematical ideas can provide insights into the nature of mathematics and into the learning and teaching of mathematics in both languages. This paper uses a typological perspective to contrast the language of comparison of isiXhosa and English, specifically when comparing countable quantities and numbers. Four canonical early grade mathematics texts in South Africa, written in English and translated into isiXhosa, were used as a source and were compared using interlinear morphemic glossing. An analysis of the texts exemplifies differences between isiXhosa and English. For example, in isiXhosa the word *ngaphezulu* can mean both "more" and "above". This can cause a conflict when using representations such as the standard hundred chart where bigger numbers appear below smaller numbers. These and other findings may be of interest to instructional designers, teacher educators, and teachers of early grade mathematics in both isiXhosa and English.

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Language of comparison; mathematical language diversity; early grade mathematics; comparison of numbers

Introduction

The comparison of languages in terms of specific mathematical ideas can provide insights into the nature of mathematics, insights for the learning and teaching of mathematics in both languages (Barton, 2008), as well as insights for curriculum development, especially for languages without a fully developed formal mathematical register (Edmonds-Wathen, Trinick, & Durand-Guerrier, 2016). For example, the means by which a language expresses a mathematical idea or process can be made explicit for teachers and learners, thus reducing confusion in multilingual classrooms (Edmonds-Wathen et al., 2016).

As the world becomes more globalised and more learners are learning mathematics in other languages (Barwell, Barton, & Setati, 2007), it becomes increasingly important to consider language differences and the pedagogical implications that these differences suggest, particularly for languages that have different grammatical structures to dominant languages such as English (Barwell, Wessel, & Parra, 2019). In this paper the language differences between English and isiXhosa, one of South Africa's indigenous

CONTACT Nicky Roberts nroberts@uj.ac.za Soweto campus, University of Johannesburg, PO Box 524 Auckland Park, 2006, South Africa

African languages, are considered by contrasting the ways in which comparisons are expressed.

The decision to compare isiXhosa[1] and English is neither neutral nor coincidental. It is a direct result of the current South African language policy environment. South Africa has 11 official languages, 9 of which are indigenous African languages. In theory, learners are able to access schooling in any of the 11 official languages and currently approximately 70% of learners are taught in an African language until the end of grade 3 (Spaull, 2016). In practice, the extent to which quality schooling is accessed remains questionable, reflecting fundamental inequalities on racial, gender, socio-economic (Roberts, 2017) and linguistic lines (Spaull, 2016).

Theoretical framing

In a recent paper, Edmonds-Wathen (2019) proposed theoretical perspectives and methodological tools from linguistics that have proved helpful for research into the diversity of mathematical expression in different languages. We have drawn from her work, both in terms of our theoretical framing and for aspects of our methodology. A typological perspective can be used in mathematics education research to study language differences, which have implications for the learning and the teaching of mathematics.

A typological perspective

Typology is an area of linguistics that describes and classifies languages according to their structural similarities and differences (Edmonds-Wathen, 2019). Typology seeks to use a framework-neutral analysis to compare languages (Nichols, 2007). Edmonds-Wathen (2019) argues that because of this neutrality, "a typological approach may be useful to investigate mathematical expression in different languages, without privileging one language over another" (p. 121). This is particularly important when comparing a language with a well-developed mathematical register (see Halliday, 1978), for example English, with a language without a formal mathematical register or with a mathematical register that is still being strengthened, for example isiXhosa.

Edmonds-Wathen (2019) illustrates the use of a typological framing by discussing the functions of prepositions in English, particularly in mathematics, and showing how other languages, such as Finnish and Pitjantjatjara, use a case system to perform the same function that prepositions perform in English. She also identifies two other topics which are of interest in mathematics education and which would be suitable for a typologically framed investigation. These are the language of modality used for reasoning[2], and the language of comparison; the latter being the focus of this study.

Language differences and learning and teaching mathematics

In terms of the implications of language differences on the learning of mathematics, we take an ecological approach to language (Moschkovich, 2017), acknowledging that while language might influence habitual thought (Lucy, 1997), and subsequently mathematical thought, there are many factors besides the structure of a learner's home language that influence their mathematical thinking (Saxe & Posner, 1983). In relation to numbers, for example, these include factors such as "the effect of schooling, tutoring, out of school experiences with numbers, cultural differences in beliefs about the importance of mathematics, and parental instruction" (Moschkovich, 2017, p. 4148).

In terms of the implications for teaching mathematics, research suggests that paying attention to language when teaching mathematics is beneficial for learners (see for example Adler, 1999 and Khisty, 1995). However, we acknowledge what Adler (1999) has termed the dilemma of transparency, namely the tension between attending to language on the one hand and not interrupting learners' mathematical thinking on the other hand. While some call for explicitly attending to features of language (e.g. Schleppegrell, 2004), others emphasise learner engagement and suggest that teachers should use strategies such as re-voicing to ensure an emphasis on meaning rather than on form (e.g. Moschkovich, 1999; Setati, 2012). There is unfortunately little empirical evidence to suggest which approach is better under which circumstances. We think that teachers need to make these decisions in their classes depending on their circumstances but that in order to be able to make informed decisions, they need to be aware of linguistic issues such as those concepts in mathematics which are expressed differently in the language of instruction and the home language of learners. One such concept, which is the focus of this study, is comparison: the identification and/or quantification of difference(s).

The language used to express comparison

The concept of comparison in early grade mathematics texts was chosen not only because of the linguistic complexity of comparisons in general but also because of the centrality of comparison in cognition and early grade mathematics.

Comparison is a basic component of cognition (Kennedy, 2009). Cognitive scientists have identified the existence of an approximate number system (ANS) allowing humans, and other animals, to distinguish between quantities (if the relative difference between the quantities is large enough) (Feigenson, Dehaene, & Spelke, 2004). The ANS is part of the core knowledge with which humans are born and research has confirmed that infants, prior to language acquisition, are able to make a distinction between quantities of many and few (Feigenson et al., 2004).

Comparison also plays a central role in teaching early grade mathematics. The National Centre for Excellence in the Teaching of Mathematics (NCETM) identifies comparison of numbers as one of the main areas of early years mathematics, differentiating it from cardinality and counting, and composition (whole-part-part concept) (NCETM, 2018a). Comparison forms the foundation of the *determining the difference* model for subtraction (Van Den Heuvel-Panhuizen & Treffers, 2009, p. 108). According to Selter, Prediger, Nührenbörger, and Hußmann (2012), the *taking away* model of subtraction is slightly more intuitive than the determining the difference model. However, when taking a longitudinal perspective on mathematical learning, being able to conceive of subtraction as determining the difference is as important, if not more important, than conceiving of subtraction as taking away.

Comparison has also been the focus of much mathematics education research, with "compare" type problems identified as one of the problem types in typologies of additive relations word problems. Roberts (2016) offers an historical development of these typologies, and Mostert (2019a) further refines the additive relation word problem types. All of the typologies of additive relation word problem types feature "compare" or "comparison" type word problems which have the common form of "compared quantity = referent ± difference".

The construction of comparisons in general

The extent to which comparison is a fundamental part of human cognition is reflected in the fact that all languages have a means of expressing gradeable concepts (e.g. big, bigger, biggest), and all languages have a means of expressing orderings between two nouns (Sapir, 1944). Comparisons can be made in terms of the degree to which two nouns possess a particular property, for example, length. Additionally, when comparing countable objects, comparisons can be made in terms of quantities.

Kennedy (2009) explains that the two linguistic components of comparisons are an ordering of superiority or inferiority and a standard against which an object is compared. Many languages use specialised morphology (e.g. words, prefixes, suffixes) and specialised syntax (e.g. rules about word order) to construct comparisons (Kennedy, 2009). English, for example, uses *more* or *-er* and *less* to establish orderings of superiority and inferiority, and *than* to identify the standard or referent against which an object is compared (Kennedy, 2009). When comparing Malagasy and Mandinka to English, Kennedy (2009) demonstrated that not all languages have specialised words (or parts of words) or specialised rules (e.g. word order) for the two linguistic components. The difference in the way in which languages express comparison raises important questions for teaching mathematics, as will be explored in this paper.

Comparison of countable quantities

When comparing two sets in terms of discrete quantities rather than continuous measurements, two additional factors come into play. The first is whether or not the difference is made explicit, that is, whether the difference is quantified. The second is whether or not the referent is stated explicitly.

Figure 1 exemplifies the difference between comparisons that quantify the difference and those that do not.

The first case, difference not quantified, does not require knowledge of the exact number of elements in each set. The second case, difference quantified, does require an understanding of exact number as well as the ability to compare two sets by means of, for example, one-to-one matching. While the first case is simpler than the second, both require the use of language and an understanding of the English words *more* and *less*. They are therefore both more complex than the comparisons infants and toddlers are able to make before they have access to language, namely distinguishing many from few (Feigenson et al., 2004).

As children are introduced to more formalised mathematics, they progress in terms of the sets they are able to compare and the ways in which they are able to compare sets. Essentially, they shift from noticing the difference to quantifying the difference. Put differently, they shift from being able to answer the question "Which is more?" to answering the question "How many more?" This is reflected, for example, in the

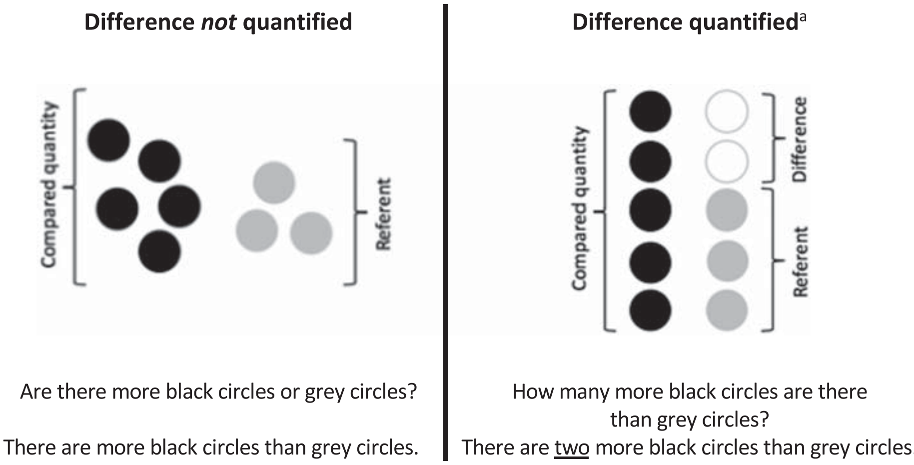

Figure 1. Comparisons with difference quantified and difference not quantified.
[a]"Difference quantified" questions are used in "comparison" type additive relation word problems (Carpenter & Moser, 1983; Riley, Greeno, & Heller, 1983).

typical progression charts of the NCETM (2018b) and in the learning trajectories of Clements and Sarama (2009).

A second factor that comes into play, especially when comparing quantities, is whether the referent is explicit or implicit. Figure 2 exemplifies the difference between comparisons that have an implicit referent and those that have an explicit referent. For phrases without an explicit referent, *more* is a determiner and can be replaced with *additional* or *another*.

	Implicit referent [more as 'additional']	**Explicit referent** [more _ than _]
Difference not quantified	Draw **more** circles.	Draw **more** circles **than** squares.[a]
Difference quantified	Draw <u>three</u> **more** circles.	Draw <u>three</u> **more** circles **than** squares.[b]

Figure 2. Differences between comparison phrases with implicit and explicit referents.
[a]Alternatively: There are more circles than squares.
[b]Alternatively: There are three more circles than squares.

These two additional factors that come into play when making comparisons in terms of quantity, namely whether the difference is quantified or not and whether the referent is explicit or implicit, form an essential component of the analysis in this paper.

Research questions

This study sets out to describe and contrast the language of comparison of English and isiXhosa, specifically as it relates to countable objects and numbers, by answering the following questions:

(1) What comparison phrases are early grade learners expected to be familiar with, according to the English version of canonical South African mathematics texts?
(2) How do English and isiXhosa comparison phrases differ, as exemplified in canonical South African mathematics texts?

Methodology

In this section we discuss the use of translated texts as source examples of the language of comparison, how these examples were analysed, and the tools used to present the data from a language different to the language of publication.

Translated texts as a source of examples

In order to compare languages using a typological perspective, examples of how each language expresses the topic under investigation are needed. Edmonds-Wathen (2019) proposes the use of structured elicitation tasks from cognitive linguistics as one source of examples. An alternative source of examples are mathematical texts and their translations. This source is particularly relevant in South Africa where the language policy framework has meant that early grade mathematics texts such as the curriculum and policy statement (CAPS) and government issued learner workbooks, originally written in English, have been translated into the other 10 official languages. This large corpus of translated mathematical texts, though sometimes flawed, provides a rich source of examples which can be used as a starting point to describe the differences in how South African languages express mathematical ideas.

Four canonical early grade mathematics texts in South Africa, written in English and translated into isiXhosa formed the basis for this study. The texts were the national curriculum document (Department of Basic Education, 2011); and three sets of learner workbooks produced by the national Department of Education (Department of Basic Education, 2018), the National Education Collaboration Trust, and the Nelson Mandela Institute. Examples were identified by searching for phrases containing the word *more* in the English version of each texts. The search was limited to the word *more* and excluded the words *less* and *fewer*. This was done in order to simplify the analysis and is a common approach in linguistic research on comparison (Kennedy, 2009). However, this is a weakness of the study as isiXhosa, like many other languages, expresses orderings of inferiority (less than) differently to the way in which orderings of superiority (more than) are expressed, as is briefly discussed in the findings.

A total of 362 phrases containing the word *more* were identified in the four sets of texts analysed. In order to simplify the analysis only contexts referring to discrete, whole number contexts were included and three types of phrases were excluded: section headings, headings in tables and labels; phrases merely listing vocabulary, for example, “Compare whole numbers using ‘smaller than’, ‘greater than’, and ‘is equal to’”; and phrases that were not considered mathematical comparison phrases, for example, “Show with an arrow more or less where this number … will be on the number line”.

Analysing examples

In order to answer the first question, the remaining 291 phrases were analysed to categorise the range of the language of comparison used in early grade English texts. Phrases were categorised according to whether the comparison was between nouns or between numbers. For both these categories phrases were further categorised according to whether the referent was explicit or implicit and whether the difference was quantified or not. After comparing the language of comparison of English and isiXhosa, this initial categorisation was further refined as described in the findings section.

In order to answer the second question, the first author compared the phrases from the English texts with those from the isiXhosa texts (which were translated from the English texts), both in general and in terms of the different categories. Existing descriptions of English and isiXhosa grammar were drawn on to depict the different ways in which the two languages expressed comparisons in general. The extent of the variation in the two languages was used to identify key components or dimensions along which the languages differed in their expression of comparison.

While the isiXhosa texts provided a valuable source of examples of how to express comparisons in isiXhosa, it was not enough to refer only to these texts. There was often much variation in how a particular type of phrase was expressed, and some phrases were incorrect, potentially because they were directly translated from English. Because the first author is not fluent in isiXhosa, as is often the case with typological research (Edmonds-Wathen, 2019), isiXhosa language experts were consulted to clarify and exemplify the range of variation possible in expressing comparisons in isiXhosa.

Presenting mathematical data in different languages

For single words or very short phrases, we follow the simple convention described by Edmonds-Wathen (2019), namely that words in the source language are presented in italics, and translations in single quotes, such as the isiXhosa word *izinja* ‘dogs’. For more extensive examples, we use a simplified interlinear morphemic gloss.[3] Interlinear morphemic glossing provides a means of presenting the structure of the example which is often lost if only an idiomatic translation is provided (Edmonds-Wathen, 2019). Interlinear morphemic glossing is particularly helpful when presenting data from languages where word order is not necessarily a determinant of the function of a word. In such languages the case or prefix of a word may denote its function.

An interlinear morphemic gloss consists of four levels (Edmonds-Wathen, 2019). For this paper the top level gives the isiXhosa in sentence form. The second level gives the isiXhosa morphemes (the smallest unit of a language that has its own meaning), the third level gives the English morphemic gloss, and the final level gives a free translation in English. Morphemes are separated by a hyphen. It is important to note that in some instances in this paper not every morpheme is glossed separately.[4] In particular, nouns and their prefixes, which indicate the noun class and the number (singular or plural) of the noun, were not glossed separately.

Findings

We present the findings in terms of the two research questions. We open with a discussion on the variability in the examples of comparison phrases in the English early mathematics texts (Question 1). The 291 comparison phrases were initially classified according to whether the comparison was between objects (countable nouns) or between numbers. For each of these categories a further distinction was made between whether the referent was implicit or explicit, and whether the difference was quantified or not. The differentiation between countable nouns and numbers reflects the shift in English from using numbers as adjectives to describe the numerosity of a collection of nouns (three pears), to using numbers as nouns themselves (three is more than five). The differentiation between the difference not being quantified and the difference being quantified exemplifies the important shift described previously, namely from *identifying* a difference to *quantifying* the difference.

This initial framework was then refined. The "countable nouns" category was separated into two categories, one in which nouns are compared (e.g. "There are more circles than stars") and one in which nouns belonging to people are compared (e.g. "Thandi has more sweets than Thembi"). Next, each category was exemplified with a statement and with a question that would prompt that statement. This provided an extensive range of the types of comparison phrases with which early grade learners are expected to be familiar. Finally, an additional subcategory was added to the *numbers [explicit referent]* category because of the difference between the phrase "8 is 3 more than 5" and "3 more than 5 is 8".

Table 1 exemplifies each of the subcategories. Note that while the canonical texts did include examples of phrases that could have been assigned to the *numbers [implicit referent]* category, such as "know which number is 3 more", these phrases were excluded because they could not be answered without making the referent explicit, thereby shifting the phrase to a different category. Also note that while the explicit referent examples in Figure 1 were phrased as commands to ensure consistency with the implicit referent examples, in Table 1 the explicit referent examples are statements. This is because the examples in Table 1 are drawn from the canonical texts where all examples of phrases with explicit referents are statements and all examples of phrases with implicit referents are commands.

In order to foreground the differences and similarities between the different subcategories, some examples in the Table 1 have been adapted (e.g. stars replaced with circles) from the four texts. The empty cells indicate a type of phrase that cannot be constructed. The examples in italics indicate that, while it is possible to construct the phrase, no examples of such a phrase occur in the canonical texts that were analysed. Although they do not occur

Table 1. Exemplification of different types of comparison phrases in English.

Category	Difference	Implicit referent [more as "additional"]	Explicit referent [more _ than _]
Nouns (objects & people)	**Not quantified**	Draw more circles.	Are there more circles or squares? There are more circles than squares.
	Quantified	*How many more circles?* Draw 3 more circles.	How many more circles are there than squares? *There are 3 more circles than squares.*
Objects (belonging to people)	**Not quantified**	*Sbu gets/needs more sweets.*	Does Sbu or Sive have more sweets? *Sbu has more sweets than Sive.*
	Quantified	How many more sweets does Sbu get/need? Sbu gets/needs 3 more sweets.	How many more sweets does Sbu have than Sive? Sbu has 3 more sweets than Sive.
Numbers	**Not quantified**		Which is more, 8 or 5? 8 is more than 5.
	Quantified		*How much more is 8 than 5?* 8 is 3 more than 5.
	Quantified		How much is 3 more than 5? 3 more than 5 is 8.

Note: Italic indicates an example that does not appear in the canonical texts

in the texts, these "missing" examples are included for the sake of completeness. In some cases, the phrases are answers to questions that occur in the texts and therefore learners are expected to be familiar with the phrases even though they do not appear in the texts.

Next we discuss the differences between English and isiXhosa comparison phrases (Question 2). We discuss the differences in terms of three broad elements: differences in terms of the construction of general comparisons (limited to comparisons with an explicit referent); differences based on whether comparisons are made in terms of properties or quantities; and the different words used in each language to express an ordering of superiority (focusing specifically on comparisons of quantities).

Relevant isiXhosa grammar

Before discussing the three elements, a few comments about isiXhosa grammar are in order. IsiXhosa, like other Bantu languages, has a noun class system, meaning that all nouns belong to a particular class as determined by the noun's prefix. As the dominant part of speech in isiXhosa, the class of a noun in a sentence determines the *form* of other parts of speech in the sentence (adjectives, verbs, pronouns etc) through a system of concordial agreement (Bryant, 2007).

IsiXhosa is also an agglutinative language with a conjunctive orthography (Pretorius, 2018). This means that different morphemes are "glued" together to form words. In isiXhosa, phonemic mutations occur when, for example, a prefix ends with a vowel and the word the prefix is being added to starts with a vowel. For example, when the prefix *kuna-* 'compared to' is added to the noun *inkwekwe* 'boy', the *a* from the end of *kuna-* and the *i* from the beginning of *inkwenkwe* coalesce to form the vowel *e* so that the word *kun<u>e</u>nkwenkwe* 'compared to the boy' is formed. The rules for phonemic mutations that are relevant for this paper are[5]: $a + a > a$, $a + i > e$, $a + ii > ee$, $a + u > o$ and $a + oo > oo$.

In isiXhosa, numbers are nouns and therefore belong to a particular class (see Mostert (2019b) for more details). When comparing numbers, it is possible to use two different forms of the number name, either the formal abstract form (*isibhozo* 'eight'), or the "personified" form (e.g. *usibhozo* 'eight'). Because these two forms have different prefixes and therefore start with different vowels, comparison phrases for numbers can appear in a number of different variations, as will be discussed in subsequent sections.

Construction of general comparisons

Here we contrast the means by which comparisons with an explicit referent are expressed in English and isiXhosa. As discussed previously, the two linguistic components of comparisons are an ordering of superiority or inferiority and a standard against which an object is compared (the referent). The third component, which is only applicable when comparing quantities, is the difference between two quantities, which can either be quantified or not. Table 2 shows the specialised words or prefixes that are used to indicate these three different components.

In terms of establishing orderings of superiority, English uses the specialised prefix *-er* or the specialised word *more*. IsiXhosa, on the other hand, does not use a specialised word or part of a word to establish an ordering. Instead, like Malagasy and many other languages, an adjective together with a referent implies an ordering of superiority, as in example (1).

In terms of indicating the referent, both English and isiXhosa use a specialised morphology. In English the word *than* is used. In isiXhosa the comparative prefix *kuna-*[6] 'compared to' is used, also seen in example (1). In some cases *kuna-* is shortened to *ku-*.[7]

(1) Intombi inde kunenkwenkwe.

intombi	*in-de*	*kune-nkwenkwe*
girl	is-tall	compared.to-boy

'The girl is taller than the boy.'

When the numerosity of two sets are compared, in isiXhosa the difference is signalled by the prefix *nga-*[8], which can be translated with 'by' in the context of comparisons as seen in example (2). In English on the other hand, the difference is not signalled by a particular word but instead the difference is inferred from its position in the sentence. Anecdotal evidence suggests that it can be difficult for isiXhosa speakers to translate phrases comparing quantities from English to isiXhosa. We conjecture that one of the reasons for this is the reliance, in English, on word order rather than on a particular word (or prefix) to signal the difference.

Table 2. Specialised words or parts of words used to express comparisons.

Components	English	isiXhosa
Ordering of superiority in terms of a property	-er / more	
Referent	than	*kuna-*[a]
Difference		*nga-*[b]

[a]The alternative forms of kuna- are kune-, kunee-, kuno- and kunoo-
[b]The alternative forms of *nga-* are *nge-*, *ngee-*, *ngo-* and *ngoo-*.

(2) Amapere maninzi kuneebhanana ngamathathu.

amapere	*ma-ninzi*	*kunee-bhanana*	*nga-mathathu*
pears	are-many	compared.to-bananas	by-three

'There are three more pears than bananas.'

English and isiXhosa differ significantly in terms of how comparisons are expressed in general. While both languages have specialised words to signify the referent, only English has a specialised word to create an ordering of superiority and only isiXhosa has a specialised prefix to signal the difference.

For isiXhosa comparison phrases, while the class of the noun being compared determines the *form* of the other parts of speech in the phrase, it does not determine the *structure* of the comparison phrase. For example, if the number of dogs (class 10) were compared, rather than the number of pears (class 6) as in (2), the structure of the comparison phrase would remain the same but a different prefix, *zi-* rather than *ma-*, would be used with the adjective *-ninzi* 'many', as shown in (3):

(3) Izinja zininzi kuneekati ngamathathu.

izinja	*zi-ninzi*	*kunee-kati*	*nga-mathathu*
dogs	are-many	compared.to-cats	by-three

'There are three more dogs than cats.'

Differences in the *structure* of comparison phrases, as discussed in the next section, depend on what type of noun is being compared (e.g. objects vs owned objects), and not the noun class of the noun that is being compared.

Differences based on what is being compared

We now discuss differences in construction in relation to the nouns being compared and the property according to which they being compared. This is done in terms of four categories. The first being (1) the comparison of objects or people in terms of a non-measurable or continuous property such as beauty or length. The remaining categories are those appearing in Table 1: (2) the comparison of sets of nouns in terms of their quantity; (3) the comparison of sets of nouns belonging to people, also in terms of their quantity; and finally (4) the special case of the comparison of numbers in which numbers act as nouns and the property being compared is their numerosity. Table 3 exemplifies the construction of comparison phrases for isiXhosa and English respectively, while Table 4 compares the different constructions.

In all four cases in isiXhosa it is possible to construct a comparison phrase in the same way. Namely, compared quantity + *is/are* +adjective + *compared to* + referent. In terms of early grade mathematics, this means that when shifting from comparing objects in terms of a property (X1 in Table 3) to comparing objects in terms of numerosity (X2) to comparing numbers (X4), the same construction is used. This is an affordance of isiXhosa that is not shared with English.

While the structure of the comparison phrases is always the same, the adjective and the prefix used with the adjective depend on the noun being compared. In particular, the adjective depends on the type of noun being compared (object vs owned object vs

Table 3. Examples of isiXhosa and English comparison phrases.

Comparison of	Case	Examples
Property	X1	Intombi intle kunenkwenkwe.
(girls, boys)		*intombi* *i-ntle* *kune-nkwenkwe*
		girl is-beautiful compared.to-boy
	E1	'The girl is more beautiful than the boy.'
Quantity	X2	Amapere maninzi kuneebhanana.
(pears, bananas)		*amapere* *ma-ninzi* *kunee-bhanana*
		pears are-many compared.to-bananas
	E2	'There are more pears than bananas.'
Quantity	X3	Amapere wentombazana maninzi kunawenkwenkwe.
(owned pears)		*amapere* *we-ntombazana* *ma-ninzi* *kuna-we-nkwenwke*
		pears of-girl are-many compared.to-of-boy
	E3	'The girl has more pears than the boy.'
Numerosity (numbers)	X4	Usibhozo mkhulu kunontlanu.
		usibhozo *m-khulu* *kuno-ntlanu*
		eight is-big compared.to-five
	E4	'Eight is more than five.'

numbers) and on the property according to which the noun is being compared, while the form of prefix used with the adjective depends on the class of the noun being compared (as shown in examples (2) and (3)).

In English, different constructions are used depending on what is being compared. The construction when comparing two things in terms of a property (E1 in Table 3) and when comparing two numbers (E4) is similar. However, in the first case (E1), *more* is used to create the comparative form of the adjective (for poly-syllabic adjectives) and in the second case (E4) *more* is the comparative adjective. This use of *more* both as a comparative adjective and to create a comparative form of a (poly-syllabic) adjective does not occur in isiXhosa.

While E1 and E4 are similar, the comparison of objects and owned objects in terms of numerosity/quantity (E2 and E3) is different. Instead of starting with the compared quantity (the pears), the construction starts with a statement of existence *there are*. This is then followed by the compared quantity and then the referent. This means when moving from comparing sets in terms of numerosity (E2), to comparing numbers (E4), the construction of the comparison phrases changes. We conjecture that this shift and the corresponding change in construction is something that teachers of English-language learners would benefit from being aware of and, potentially, making explicit to their learners.

Differences in terms of words used to express an ordering of superiority

In this section both comparisons with an implicit referent (e.g. Draw more circles) and those with an explicit referent (e.g. There are more circles than squares) for discrete

Table 4. Differences in structure of English and isiXhosa comparison phrases.

Case	Comparison of	English structure					isiXhosa structure translated		
1	Property (nouns)	[A]	is	more	adj	than [B].	[A]	is-adj	compared to [B].
2	Quantity (nouns)	There	are	more	[A]	than [B].	[A]	are-many	compared to [B].
3	Quantity (owned nouns)	[A]	has	more	noun	than [B].	[A]	are-many	compared to [B].
4	Property (numbers)	[A]	is	more		than [B].	[A]	is-big	compared to [B].

objects are considered. In particular, phrases in the English and isiXhosa early grade texts are analysed in terms of the words used to express an ordering of superiority (in terms of quantity). Table 5 provides an overview of the different words used in both languages in three categories. Note that, as mentioned previously, it is not possible to construct a comparison phrase with an implicit referent when working with numbers.

In the English texts, the word *more* is used in all three categories. The only exception is the use of *greater* when comparing numbers. While they were not used in the texts, alternative words for greater that are commonly used in English when comparing numbers are *larger* and *bigger*.

In isiXhosa, a wider range of words is used. We discuss these in terms of different parts of speech that are used: adjectives; adverbs; verbs and words implying "more".

Adjectives

In comparison phrases with an explicit referent, the more commonly used isiXhosa construction consists of an adjective plus a referent to express an ordering of superiority. These phrases therefore do not have the equivalent of the English word "more". When comparing objects, the adjective is *-ninzi* 'many' or 'numerous' and when comparing numbers, the adjective is *-khulu* 'big'. In using the adjective *-khulu* 'big' when comparing numbers, isiXhosa mirrors the English use of *greater*, *bigger* and *larger* when comparing numbers. In both cases numbers are compared in terms of a property (their size), rather than in terms of their numerosity.

In comparison phrases with an implicit referent (e.g. "Draw more circles"), isiXhosa also makes uses of an adjective, namely *-nye*. When used with a singular prefix (e.g. *esinye*) *-nye* can mean both 'one' and 'another one' or 'an additional one' (depending on whether a low or a high tone is used). When used with a plural prefix (e.g. *ezinye*), it means 'some others' or 'additional'. All three adjectives (*-nye, -ninzi, and -khulu*) can be used when the difference is quantified and when it is not quantified (see Table 6 for examples).

Adverbs

Even though it is possible to construct isiXhosa comparison phrases without the use of a word like the English word *more*, isiXhosa does have a word, *ngaphezulu*, which can be used in a similar manner to the English word *more*. *Ngaphezulu* is the locative adverb[9] of *phezulu* 'on top' or 'above' which is related to the noun *izulu* 'heaven' (Oosthuysen, 2016, p. 43). *Ngaphezulu* can refer to the spatial relationship between two objects, namely that

Table 5. Words used to express an ordering of superiority in English and in isiXhosa.

Category	English	Part of speech	isiXhosa	Direct translation	Part of speech
Implicit referent (objects)	more	DETERMINER	*-nye*[a]	'additional'	ADJECTIVE
Draw C more.			*-ngaphezulu*	'more'	ADVERB
			-ongeza	'add'	VERB
Explicit referent (objects)	more	ADJECTIVE	*-ninzi*	'many'	ADJECTIVE
A is more than B.			*-ngaphezulu*	'more'	ADVERB
			-dlula	'exceed'	VERB
Explicit referent (numbers)	greater	ADJECTIVE	*-khulu*	'big'	ADJECTIVE
A is more than B.	more	ADJECTIVE	*-ngaphezulu*	'more'	ADVERB

[a]This is the same word as for the number one.

Table 6. Adjectives used in isiXhosa both when the difference is quantified and when it is not quantified.

Category	Adjective	Difference not quantified	Difference quantified
Implicit referent (objects)	*-nye* 'another'	Zoba ezinye izangqa. *zoba ezinye izangqa* draw some.other circles 'Draw more circles.'	Zoba ezinye izangqa ezintathu. *zoba ezinye izangqa ezi-ntathu* draw some.other circles that.are-three 'Draw three more circles.'
Explicit referent (objects)	*-ninzi* 'many'	Amapere maninzi kuneebhanana. *amapere ma-ninzi kunee-bhanana* pears are-many compared.to-bananas 'There are more pears than bananas.'	Amapere maninzi kuneebhanana ngamathathu. *amapere ma-ninzi kunee-bhanana nga-mathathu* pears are-many compared.to-bananas by-three 'There are three more pears than bananas.'
Explicit referent (numbers)	*-khulu* 'big'	Usibhozo mkhulu kunontlanu[a]. *usibhozo m-khulu kuno-ntlanu* eight is-big compared.to-five 'Eight is bigger than five.'	Usibhozo mkhulu kunontlanu ngontathu. *usibhozo m-khulu kuno-ntlanu ngo-ntathu* eight is-big compared.to-five by-three 'Eight is three bigger than five.'

[a]An alternative, more abstract, construction would be: Isibhozo sikhulu kunesihlanu.

one object is above or on top of another object; or *ngaphezulu* can be used to express an ordering of superiority, namely that one quantity is more than another.

Because *ngaphezulu* can be used in two ways, there is potential for ambiguity. It is, however, usually clear from the context and the concords (prefixes) whether *ngaphezulu* means 'above' or 'more'. For example, when the difference is quantified, *ngaphezulu* always means 'more'. When the referent is implicit and the difference is not quantified, *ngaphezulu* means 'above'. When objects are compared and the referent is explicit but the difference is not quantified, different concords (prefixes) are used with the referent when the objects are compared in terms of numerosity (*kuna-*[10] or the shortened *ku-*) and when the objects are compared in terms of the spatial relationship between them (*kwa-*[11]). This means that *-ngaphezulu kuna-* can be translated as 'more than' and *-ngaphezulu kwa-* can be translated as 'above'. Note that it does not make sense to use the concord *kwa-* when the difference is quantified.

Note that even though *-ngaphezulu kwa-* strictly speaking means 'above', when comparing numbers *-ngaphezulu kwa-* is taken to mean 'more than'. This is because numbers are almost exclusively compared in terms of the difference in their numerosity and not in terms of their spatial relationship. Therefore, both *kuno-* and *kwa-* are used to indicate 'more than' although *kuno-* and the shortened version *ku-* are more commonly used. See Table 7 for examples of *ngaphezulu* used in comparison phrases.

In spoken language *ngaphezulu* is often shortened to *ngaphezu*. The variation possible in terms of the prefix used to indicate the referent, as well as the variation possible in terms of the prefix used for number names, together with the rules for coalescence means that there are up to six different ways to construct the comparison of two numbers, some more commonly used that others. The isiXhosa translations analysed in this study show that a wide range of these possible variations are used. Whether the form of the number names (abstract or personified) or the prefix used to indicate the referent (*kuna-*, *ku-* or *kwa-*) make a difference in terms of the ease with which learners make sense of comparisons of numbers requires further research.

The use of the isiXhosa word *ngaphezulu* ('above' or 'more') has two potential implications for teaching mathematics in isiXhosa. The first is a "directional conflict"

Table 7. The use of *-ngaphezulu* in isiXhosa phrases.

Category	Difference not quantified	Difference quantified
Implicit referent (objects)	Zoba izangqa ngaphezulu. *zoba izangqa ngaphezulu* draw circles above 'Draw circles above.'	Zoba izangqa ezingaphezulu ngezintathu. *zoba izangqa ezi-ngaphezulu nge-zintathu* draw circles that.are-more by-three 'Draw three more circles.'
Explicit referent (objects)	Amapere angaphezulu kuneebhanana. *amapere a-ngaphezulu **kunee**-bhanana* pears are-more compared.to-bananas 'There are more pears than bananas.'	Amapere angaphezulu kuneebhanana ngamathathu. *amapere a-ngaphezulu **kunee**-bhanana nga-mathathu* pears are-more compared.to-bananas by-three 'There are three more pears than bananas.'
	Amapere angaphezulu kweebhanana. *amapere a-ngaphezulu **kwee**-bhanana* pears are-above in.relation.to-bananas 'The pears are above the bananas.'	
Explicit referent (numbers)	Usibhozo ungaphezulu kunontlanu[a]. *usibhozo u-ngaphezulu kuno-ntlanu* eight is-more compared.to-five 'Eight is more than five.'	Usibhozo ungaphezulu kunontlanu ngontathu. *usibhozo u-ngaphezulu kuno-ntlanu ngo-ntathu* eight is-more compared.to-five by-three 'Eight is three more than five.'

[a]This is one of six possible constructions for 'more than five'. Another is: ungaphezulu kwisihlanu.

(Randolph & Jeffers, 1974) which occurs when using isiXhosa to describe the relationship between numbers on a standard hundred chart (where bigger numbers are displayed below smaller numbers). The word used to describe the *quantitative* relationship between the numbers, namely *ngaphezulu* 'more' (or 'above'), is the opposite of the word used to describe the *spatial* or *directional* relationship between the numbers, namely *ngaphantsi* 'below' (or 'less'). While this conflict has also been raised as a concern in English (Bay-Williams & Fletcher, 2017) where *more* and the value of a number *going up* are conceptually linked, in isiXhosa this conflict is more pronounced as exactly the same word is used for *more* and *above*. In both languages this conflict would be resolved if a bottom-up hundred chart (which displays bigger numbers above smaller numbers) was to be used. In fact, in isiXhosa, a bottom-up hundred chart would allow teachers to leverage the relationship between both the spatial and quantitative meanings of *ngaphezulu* with the spatial and quantitative relationship between numbers on the number chart (with bigger numbers being *above* smaller numbers). While there is no empirical research demonstrating the benefits of using such a chart, Bay-Williams and Fletcher (2017) provide some anecdotal evidence and list potential benefits. They also emphasise that while this can be beneficial for all learners, in particular it can avoid confusion for English-language learners.

The second pedagogical implication also relates to leveraging the relationship between the two meanings of *ngaphezulu* and vertical representations of numbers. When representing numbers using vertical cube towers (rather than horizontal cube trains), if a number is *ngaphezulu* 'more' than another number it is also, in a sense, *ngaphezulu* 'above' the other number. Similarly, if numbers are represented on a vertical number line, a number that is *ngaphezulu* 'more' than another number is also *ngaphezulu* 'above' the other number. This alignment between language and representation could be leveraged by teachers. While such conjectures need to be tested empirically, we suggest that emphasising vertical representations, especially the use of towers when initially comparing numbers, could prove beneficial for isiXhosa learners.

Verbs

There are two verbs used to express “more” in the isiXhosa early grade texts. The first is the verb *-ongeza* ‘add’ which is only used with an implicit referent as in example (4).

(4) Umama wongeza ilekese enye.

umama	*wo-ngeza*	*ilekese*	*enye*
mother	she-adds	sweet	one

‘Mother gives one more sweet.’

The second is the verb *-dlula* ‘exceed’ which is only used with an explicit referent in the texts as in example (5). While it would be possible to use the word *exceed* in English comparison phrases, it is not used in the early grade texts.

(5) UNosisi wodlula uThemba ngeebhanana ezingaphi?

unosisi	*wo-dlula*	*uthemba*	*ngee-bhanana*	*ezi-ngaphi?*
Nosisi	hers-exceed	Themba	by.means.of-bananas	that.are-how.many?

‘How many more bananas does Nosisi have than Themba?’

Words implying “more”

“Reach a target” word problems[12] such as “Simphiwe wants ten hens on his farm. He has four hens. How many more hens must he get to have ten?” are a special case of change type additive relation word problems. In isiXhosa reach a target word problems, the word *-kusafuneka* ‘it still requires’ is often used in the question instead of using a specific word for “more”, as in example (6).

(6) Kusafuneka afumane ezingaphi ukuze abe nezilishumi?

ku-sa-funeka	*a-fumane*	*ezi-ngaphi*	*ukuze*	*abe*	*ne-zilishumi?*
it-still-requires	him-to.get	that.are-how.many	to.be	there	with-ten?

‘How many hens does he still need to have ten?’

In other word problems, where an action results in an increase in the number of objects, a particular word can be used to imply “more”, such as *-fika* ‘arrive’ in example (7).

(7) Bekukho amabhabhathane amathandathu. Kwafika amabini.

be-kukho	*amabhabhathane*	*ama-thandathu.*	*kwa-fika*	*ama-bini*
were-there	butterflies	that.are-six	there-arrive	that.are-two

‘There were six butterflies. Two more arrived.’

The previous examples show that isiXhosa texts make use of a wider range of words to express an increase in number, than English. In the English text, the word *more* is almost exclusively used even though there are other words that could be used, for example *additional* (e.g. “Draw an additional three circles”) or *exceeds* (e.g. “Sbu’s marbles exceed Sive’s by three”).

Discussion

While these findings point to pedagogical approaches that might prove effective in isiXhosa dominant classrooms, they also suggest ways to support English-language learners, both with isiXhosa as a home language and with other languages as a home language. This is important because, with the increase in linguistic diversity in mathematics classrooms worldwide, teachers of mathematics in English are likely to have to support English-language learners and to have the opportunity to proactively build on children's home language resources. Approaches to supporting learners engaging with mathematics in an additional language, would benefit from teachers being aware of both the potential areas of difficulty, as well as the potential strengths that learners bring.

We therefore argue that teachers of mathematics in English would benefit from knowledge of ways in which comparison may be expressed in languages other than English. Such knowledge may be built by comparing English to languages which do not share its linguistic structures (such as isiXhosa). Examining the way in which comparison is expressed in early grade English and isiXhosa mathematical texts, reveals several features of how comparison is expressed. We think that if teachers of mathematics in English are aware of these features, they may be able to reduce confusion for English-language learners, or use their home language as a resource for learning mathematics.

Building on Kennedy (2009), we note that *general comparisons consist of two components: an ordering of superiority or inferiority and a standard against which an object is compared, also known as the referent (feature 1)*. Mathematics teachers would benefit from knowing that *some languages use a specialised word to express an ordering of superiority while in other languages the ordering is inferred by the presence of a referent (feature 2)*. In English, the ordering of superiority is expressed by the comparative adjective created by adding the suffix *-er* or the word *more* (for poly-syllabic adjectives). In English *more* is also used as a comparative adjective when comparing numbers. We turn now to the referent. In English the referent is always preceded by the word *than*. In isiXhosa the referent is preceded by *kuna-* 'compared to'. Attending to the referent is important as *how comparison is expressed in English depends on whether the referent is implicit or explicit (feature 3)*.

Further linguistic complexity is introduced, in English, depending on what is being compared, as summarised in Table 4. We suggest that mathematics teachers *be deliberate about comparing nouns, comparing objects belonging to people, or comparing number names (feature 5)*. Table 1 summarises the various ways in which comparison is expressed in English in mathematics texts, and may serve as a guide to teachers seeking to ensure that they offer children exposure to the relevant linguistic diversity in mathematics texts. We conjecture that English-language learners with a home language that is able to express comparisons in the same way, irrespective of what is being compared, in particular, would benefit from engaging with the different constructions used for comparison in English.

"The difference" is another component of comparison phrases although it is only applicable when comparisons are made in terms of quantity or numerosity. *When comparing objects in terms of quantity there are three components to consider: the compared quantity, the referent and the difference (feature 6)*. We argue that it would benefit mathematics teachers to *be aware of the distinction between situations where the difference is not quantified, and situations where the difference is quantified (feature 7).* This distinction has pedagogical implications as children take time to progress from the former to the latter when interpreting and expressing comparison, the latter being used in typical compare word problems. Figure 1 provides a visualisation of the two types of comparison situations (difference not quantified and difference quantified) and the three relevant components.

In our analysis we highlighted the isiXhosa affordance of utilising the prefix *nga-* to make the difference explicit. This affordance can be leveraged when teaching mathematics in English, both to English learners and to English-language learners. Rather than using statements like "8 is 3 more than 5", teachers of mathematics in English could *make the difference in comparative statements more explicit by (at least initially) utilising a less common phrasing of "8 is more than 5 by 3" (feature 8).*

Finally, mathematics teachers would benefit from being sensitive to the fact that *in some languages (such as isiXhosa) the order of superiority may be expressed using a locative adverb (feature 9),* for example when an ordering of superiority is expressed by a locative adverb also meaning 'above'. This demonstrates that even though mathematical language might require the use of particular linguistic features, different languages may make use of different linguistic resources to express the same mathematical idea (Edmonds-Wathen, 2019). It is therefore also important for mathematics teachers to be sensitive to the possible directional conflict this could result in when using the standard hundred chart and to ensure that they use vertical representations of numbers (e.g. cube towers and vertical number lines) and not only horizontal representations.

Conclusion

This paper provides an example of how the differences between two languages, English and isiXhosa, can be analysed in terms of a specific mathematical concept, namely comparison. The analysis has revealed important salient features of both languages relevant to comparison. In as much as these features have implications for learning and teaching mathematics, the findings are valuable for those teaching mathematics in either language, and especially for those teaching mathematics in one language to learners who have the other languages as a home language.

The differences between English and isiXhosa, which are likely to occur in other languages that do not use a specialised word for "more" when expressing comparisons in general, have important implications for teaching the early grade mathematics concept of comparison, and for creating or translating early grade mathematics texts. While this paper did not set out to measure the effect of these differences on learner performance, it had laid the groundwork for such further research.

Using a typological approach to describe the language in which the mathematics is expressed as a dimension of possible variation, deepened our reflection on the mathematical concept of comparison. By keeping the mathematical concept invariant, but changing the language in which the concept is expressed, we see the mathematical concept in a new way and become aware of the constraints imposed as well as the affordances enabled when working mathematically in a particular language. Having examples of languages with different linguistic structures is therefore a powerful resource for thinking about particular mathematical concepts. Rather than assuming that the linguistic diversity in a mathematics classroom is a challenge to be overcome, it can be viewed as an opportunity to be harnessed for supporting children in learning mathematics.

Notes

1. isiXhosa is one of four Nguni languages spoken in South Africa, the others being isiZulu, isiNdebele and siSwati. isiXhosa is spoken by more than 8 million South Africans (of a total of 57 million). As a Bantu language, isiXhosa has many linguistic features that differ substantially from the linguistic features of English and other Indo-European languages.
2. Modality refers to linguistic devices that indicate the degree to which an observation is possible, probable, likely, certain, permitted, or prohibited. For example “Sive must be home” and “Sive might be home” have different modal meanings.
3. See Comrie, Haspelmath, and Bickel (2008) for the Leipzig glossing rules used widely in linguistics
4. IsiXhosa is an agglutinative language and words are made up of many morphemes.
5. In linguistics the > sign signals that two letters have combined and are replaced by one new letter.
6. The prefix *kuna-* is made up of the locative prefix *ku-* and the connective prefix *na-* and should technically be glossed as *ku-na* “by-with”. However, for the purposes of this paper *kuna-* will be glossed as “compared.to”.
7. The alternative forms of *ku-* are *kwi-, kwii-,* and *koo-*
8. *nga-* is the instrumentative prefix used to describe an instrument e.g. She wrote in the sand *by means of (with) her finger*.
9. A locative adverb is a type of adverb that refers to a location or to a combination of a location and a relation to that location
10. The alternative forms of *kuna-* are *kune-*, *kunee-*, *kuno-* and *kunoo-*
11. The alternative forms of *kwa-* are *kwe-*, *kwee-*, *ko-*, *koo-*
12. We distinguish between “reach a target” problems which involve one set and “equalise” problems which involve two sets. While the early work on additive relation word problems included equalise problems (see Carpenter and Moser (1983) as well as Riley, Greeno, and Heller (1983)), later work generally excludes this category. See Mostert (2019a) for a more detailed discussion about equalise problems.

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ORCID

Ingrid Mostert http://orcid.org/0000-0001-9756-6622
Nicky Roberts http://orcid.org/0000-0002-1910-0162

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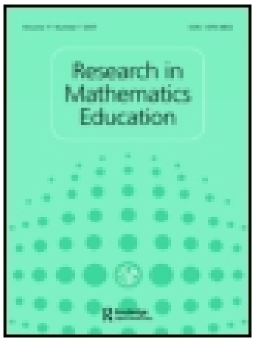

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Susan Staats & Claire Halpert

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COMMENTARY

Comparisons and their magic: a commentary on "Diversity of mathematical expression: the language of comparison in English and isiXhosa early grade mathematics texts" by Ingrid Mostert and Nicky Roberts

Susan Staats and Claire Halpert

University of Minnesota, USA

> On difficult subjects I have given several texts referring to the same word. (Malinowski, 1965/1935, p. 4)

We are delighted to contribute a text about Mostert and Roberts' text about isiXhosa means of expressing numerical comparisons. Their use of syntactically-focused linguistics methods contributes to a growing area of mathematics education research. For example, corpus linguistics methods tracked the circulation of authority within classrooms (Herbel-Eisenmann & Wagner, 2010). Structured elicitation techniques revealed grammaticalized spatial repertoires among First Nations peoples of Australia (Edmonds-Wathen, 2019). As a mathematics teacher, Lunney Borden used perhaps the most direct application of a linguistics method, to learn a language from people in your community because you need to (2018; see also Barton, Fairhall, & Trinick, 1998). In class, she noticed that her Mi'kmaw students did not usually speak of the "flatness" of the face of a cube, but rather, that the cube could "sit still". Subsequently, her verb-based questions, "How does the line change?" rather than "What is the slope?" deepened students' mathematical discussions (2018). Lunney Borden's intentional, syntactic translanguaging pedagogy based on the verbification of the English mathematics register remains a rare classroom realisation of syntactic awareness.

Mostert and Roberts' paper extends these efforts with an admirably multileveled analysis of isiXhosa curricular translations, situated within research on early childhood number cognition, mathematical discourse research, and sociopolitical conditions of mathematics education in South Africa. They validated and extended their use of linguistics scholarship through direct engagement with isiXhosa-speaking colleagues. Further, Mostert and Roberts took the difficult step of imagining potential educational applications of syntactic analysis. They critique a commonplace mathematical representation based on isiXhosa spatial repertoires. They anticipate that the consistent structure for comparisons may complicate isiXhosa speakers' interpretation of more varied structures in English (Table 4). Mostert and Roberts also suggest that English-speaking teachers could translanguage using isiXhosa syntax as in *"8 is more than 5 by 3"* (p. 33), a suggestion that evokes Lunney Borden's use of Mi'kmaq verbification.

Mostert and Roberts' careful attention to isiXhosa syntactic detail is conveyed through "interlinear glosses" in which multiple lines are used to document the original sentence, grammatical information, and a free translation into the language of publication, in this case, English. Edmonds-Wathen describes this translation approach, the Leipzig Glossing Convention, in her 2019 paper in *Research in Mathematics Education*. Interdisciplinary researchers encounter the dilemma of the degree of technical precision to introduce into each of their papers. Mostert and Roberts use a simplified version of the interlinear glossing convention, while Edmonds-Wathen uses a more detailed version which documents the morphemes, or small, meaningful components of words which expose syntactic differences between languages. In our example below, we have used the more detailed glossing approach. Both papers take a tremendously important step forward in mathematics education research by acknowledging that translation into English may not neutrally capture the original meanings and resonances of a sentence.

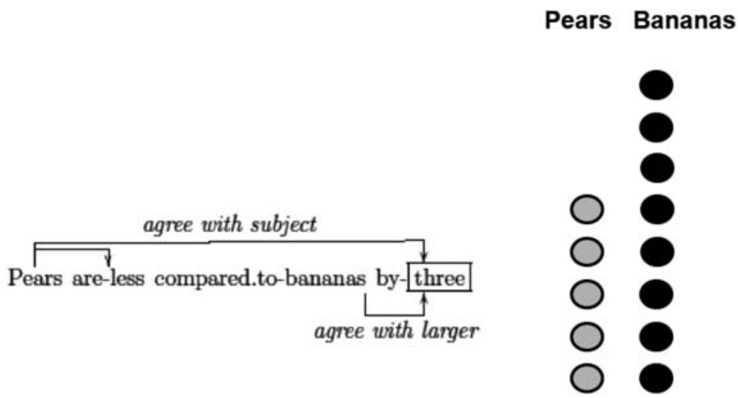
Mostert and Roberts' use of interlinear glosses reveals a number of interesting grammatical properties and suggests new directions for syntactically-based teaching and learning research. For example, isiXhosa-speaking teachers could use a revoicing strategy based on the comparison structure highlighted in the paper to help children transverse the learning progression of noticing a difference to quantifying it.

In terms of the glossed presentation, the examples given show a characteristic syntactic feature of Bantu languages, the noun class (a categorisation of nouns). Mostert and Roberts note that noun classes, in comparison phrases, determine the *form* of the other parts of speech. In Bantu linguistics, "pear" is annotated as belonging to noun class 6 when it is plural, using the **(a)-*ma*-** marker; plural "banana" belongs to noun class 8. Our translation (below) includes the noun class number and other grammatical notations such as the subject marker (SM); comparison markers which (as Mostert and Roberts explain) involve locative (LOC) and adjective (ADJ) morphemes.

(1) Students:	Amapere	maninzi	kuneebhanana.	
	Ama-pere	*ma-ninzi*	*ku-nee-bhanana.*	
	6-pear	6SM-many	LOC-with.8-banana	
	'There are more pears compared to bananas.'			
Teacher:	Amapere	maninzi	kuneebhanana	ngama-_______?
	Ama-pere	*ma-ninzi*	*ku-nee-bhanana*	*nga-ma-______?*
	6-pear	6SM-many	LOC-with.8-banana	by.6-6ADJ-___?
	'There are more pears compared to bananas by __?'			

We suggest that extending the interlinear morphemic annotation, particularly of isiXhosa noun class markers, would facilitate future research. Without annotation of the noun class, the glosses obscure a potentially important feature. When comparisons are quantified, the number that quantifies the difference (marked with *nga* "by") also appears to change form – or agree – with the noun being compared (see Table 6). When ***ama**-pere* "pears" are being compared, the number *ng-**ama**-thathu* "three" matches; when ***u**-sibhozo* "eight" is being compared, the number *ng-**o**-ntathu* "three" matches that instead. Grammatically, it looks like the number quantifying the difference "belongs" to the object compared, based on those examples. The added detail to the glosses in our example above highlights this pattern: in the teacher response, whatever number students fill in can be expected to be marked by the class 6 adjective prefix *a-ma-*.

This grammatical point invites a question for the further study of "less than" comparisons using positive integers. We expect that "pears" will agree with the predicate or verb phrase "are-less." But does the quantified difference "belong to" the subject of the sentence – here, the smaller number – in every case, or does it match whichever number is larger, regardless of position in the sentence?



Suppose that a student explains an image of five pears (class 6) and eight bananas (class 8) using the "less than" version of the comparison statement. If "three" agrees with, or belongs to, "pears", then this repetition of the noun class 6 marker emphasises a quantity that is not visually available, bringing it into palpable, though auditory, existence. If "three" belongs to "bananas," then the repetition of the noun class 8 marker emphasises a visible quantity. Both cases reinforce the broader syntax of comparison, and could possibly lead to scaffolded teaching pathways for partitioning integers, but with different starting points. We hope this example indicates the value of full annotation in the interlinear gloss, including the syntactic information such as the noun class, for multilingual examples.

Mostert and Roberts' cross-linguistic comparison of the syntax of mathematical language, along with our speculative example, necessarily raises the question of linguistic relativity, whether the lexical and syntactic systems of languages shape speakers' construals of the world. Mostert and Roberts' appropriately measured position acknowledges multiple influences on students' school-based learning and avoids positioning the researchers' primary language as the measure of others (Moschkovich, 2013). An enduring lesson from linguistic relativity research is that merely identifying syntactic differences does not necessarily imply differences in thought (Lucy, 2016). Contrary to Orr's famously criticised study of African American Vernacular English (1997), students who use a comparison such as "twice as less" are not expressing a cognitive difference, and they will not resolve the problems of structural racism in their country by aligning their grammars to those commonly used by their White teachers. Methodologically, linguistic relativity studies usually protect against ungrounded interpretation by gathering extra-linguistic evidence, often experimentally. Alternatively, a recent anthropologically-informed approach accomplishes this through analysis of the structure of spoken interaction, noting that differing syntaxes make available differing means of taking social action (Sidnell & Enfield, 2012). The idea that language constitutes social action traces to Malinowski's analysis of Kiriwina gardening magic; Sidnell and Enfield's contribution shows that syntax, within its "extralinguistic" context of gesture, intonation, pause, and rhetorical structure, that spans multiple sentences and speakers, is sometimes at the centre of the invisible magic, whether unifying or disruptive, of sociality.

Thus far, syntactically-focused mathematics education research has envisioned new classroom speaking strategies, critique of ethnolinguistic biases of mathematical representations, the negotiation of classroom authority, and translanguaging pedagogies. But with these opportunities come the hazards of comparison (Moschkovich, 2013). Without deep understanding of the politics of language use, translanguaging can devolve into cognitive reification (Orr, 1997) or prejudicial mimicry (Hill, 2007). Following Sidnell and Enfield (2012), multimodal analysis collating syntax, gesture, prosody and rhetorical structure could critically investigate whether or not syntax influences mathematical conceptualisations, for example, in the noun class agreement example above. With these considerations in mind, we look forward to further comparisons of the methods of linguistics and mathematics education through syntactically-aware research.

Disclosure statement

No potential conflict of interest was reported by the authors.

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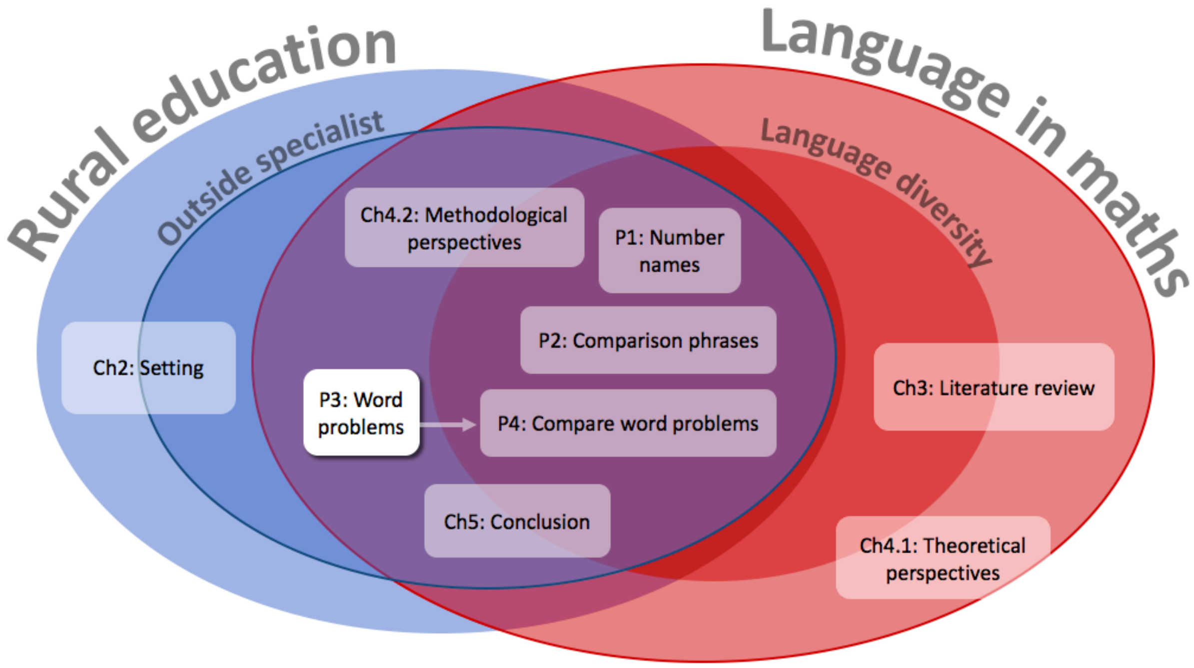
PAPER 3: Distribution of additive relation word problems in South African early grade mathematics workbooks



The third paper at the core of this study was prompted by observing an over reliance in rural classrooms on DBE workbooks; and a concern about the implications of this, particularly as it relates to learners' exposure to additive relation word problems. Unlike the first two papers, this paper does not fall within the domain of language diversity as it only considers one language, namely English. However, because it considers additive relation word problems, it still falls within the broader domain of language in mathematics. Like the first two papers, the third paper also falls in the domain of rural education research, as undertaken by an outsider. However, unlike the first two papers, the third paper relates more to the instructional practices of rural schools and less to their linguistic profile. The unit of analysis of the third paper is the collection of additive relation word problems in the English version of the DBE workbooks.

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Distribution of additive relation word problems in South African early grade Mathematics workbooks

Author:
Ingrid Mostert[1]

Affiliation:
[1]Centre for Education Practice Research, University of Johannesburg, Soweto, South Africa

Corresponding author:
Ingrid Mostert,
ingridmostert@gmail.com

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Background: Workbooks were introduced by the South African Department of Basic Education (DBE) in 2011. Although the workbooks were designed as supplementary materials, in some schools they are used as the sole teaching text. Therefore, an analysis of the content coverage of the workbooks is warranted. This article provides such an analysis in terms of additive relation word problems.

Aim: This article aims firstly to expound on the existing literature to propose a comprehensive additive relation word problem typology and secondly to analyse the prevalence of particular word problem types in the foundation phase Mathematics workbooks.

Setting: This research was conducted in South Africa, focusing on additive relation word problems in foundation phase Mathematics workbooks.

Methods: A comprehensive typology of additive relation word problem types was developed based on typologies used in previous studies. All the additive relation word problems in the 2017 Grades 1–3 foundation phase Mathematics workbooks were categorised according to this typology.

Results: In total there were 61 single-step additive relation word problems with numerical answers across the three grades. This is a small number in comparison to other countries. There was also an uneven distribution of problem types, with more problems in the easier subcategories and fewer or no problems in the more difficult subcategories.

Conclusion: This article provides evidence for the need to revise the word problems in the DBE workbooks. It also provides a theoretical framework to use in the revision of the workbooks and in any supplementary teaching material developed for teachers.

Introduction

Since 2011 the South African Department of Basic Education (DBE) has provided all Grade 1–6 learners in public schools with Language, Mathematics and Life Skills workbooks. The intention of the workbooks, as explained on the DBE website (DBE 2018), is that they 'provide every learner with worksheets to practise the language and numeracy skills they have been taught in class'. This implies that the workbooks are not to be used as a teaching resource but as a supplementary resource. They are not meant to replace textbooks but rather to replace the need for teachers to create their own worksheets. Designing the workbooks in this way both acknowledges what the 2012 National Education Evaluation and Development Unit (NEEDU) report (Taylor 2013:44) describes as South Africa's 'ubiquitous worksheet culture' and addresses the concern raised by the report, namely that only a minority of teachers are capable of creating a well-developed set of worksheets.

Although the intention is that the workbooks do not replace textbooks, Hoadley and Galant (2016) refer to recent calls to trial the workbooks as the sole text given to teachers. Anecdotal evidence from the Eastern Cape suggests that there are many schools, particulary text-poor schools, where workbooks are already the sole teaching resource. This is because, while most schools do not have sufficient textbooks, they do have workbooks 'in the right quantities and correct language' (NEEDU 2016, slide 10). Not only do schools have workbooks, but teachers have generally taken up the workbooks positively (Taylor 2013).

Should it be true that teachers use workbooks as their sole teaching resource, it implies that learners will only see the examples and exercises that appear in the workbooks and will not be exposed to any other examples. It is therefore important that variety and frequency of problems is given considerable attention. Even if it is not the case that a large number of teachers rely solely on the workbooks, the workbooks still represent an exemplification, by the DBE, of the curriculum and therefore warrant careful and detailed examination in terms of coverage of the prescribed curriculum.

This study provides such an analysis in relation to one aspect of foundation phase Mathematics, namely word problems. As such, it sets out to contribute to the small amount of literature on foundation phase Mathematics workbooks. This literature includes the work of Mathews, Mdluli and Ramsingh (2014) on how teachers use the workbooks and of Davis (2012) and Galant (2013), who point out misconceptions in the workbooks about addition and multiplication, respectively.

Word problems are a key aspect of early grade Mathematics curricula, internationally and in South Africa (Department of Basic Education 2011). Word problems are linguistically expressed Mathematics problems and therefore have implications for vocabulary, syntax and even morphology (Halliday & Matthiessen 2013). Three diagnostic reports on the South African Annual National Assessments (ANA) have identified word problems as a recurring weakness (Department of Basic Education 2012, 2014, 2015) in the foundation phase, thus warranting further research. Although the ANA is no longer an annual part of the South African basic education landscape, these standardised assessments provide useful data around curriculum expectations at the level of classroom assessment practices.

In Grades 1–3, word problems focus on additive relations and multiplicative reasoning. Various Mathematics education researchers have invested time in classifying additive relation word problems and developing typologies against which a curriculum or text could be evaluated. In schools where workbooks are the sole teaching text, the only examples of word problems that learners see are those in the workbooks. It is therefore both important and practically relevant to reflect on the extent to which South Africa's Mathematics workbooks in Grades 1–3 cover the different word problem types and how frequently different types of problems are posed.

Although similar studies have been done in a number of different countries, such an analysis has not been done on South African foundation phase textbooks or the workbooks. Stigler et al. (1986) compared US textbooks to Soviet textbooks and found that both textbooks had between 300 and 400 word problems across the first three grades. However, while US textbooks over-represented some problem types and under-represented others, Soviet textbooks had a more even distribution of problem types. More recently, a number of studies have shown that the distribution of word problems in other countries such as Turkey (Olkun & Toluk 2006; Tarim 2017), Malaysia (Parmjit, Sahari & Moideen 2006), Brunei (Dhindsa, Veloo & Singh 2014) and Greece (Despina & Harikleia 2014) is similar to the uneven distribution in the US textbooks. This study will establish whether South Africa's foundation phase workbooks also have a similar uneven distribution of word problem types or whether they are more evenly distributed as in the Soviet Union in 1986.

This study will not only differ from other similar studies in terms of its focus on South Africa, it will also differ in terms of the level of attention paid to the typology used in the analysis. While other studies have adopted (or slightly adapted) an existing typology, this study describes alternative typologies and explains why they were not chosen. It also argues for the renaming of certain problem types (building on the work by Roberts [2016a]) and extends the chosen typology by including variations of certain problem types.

The renamed, extended and exemplified typology that is developed in this article provides not only an analysis tool for the current workbooks but also a practical reference tool for curriculum developers, workbook and textbook writers and teachers – both in South Africa and internationally – to make informed decisions about the types and spread of word problems offered to learners. In South Africa such a tool is valuable in the light of the problematic typologies uncovered by Roberts (2016a) in the South African Curriculum and Assessment Policy Standard (CAPS). It is also potentially valuable should the foundation phase Mathematics workbooks be reworked.

It is for these reasons that this article aims firstly to expound on the existing literature to propose a comprehensive additive relation word problem typology and secondly to analyse the prevalence or absence of particular word problem types in the DBE foundation phase Mathematics workbooks.

Theoretical framework

Early typologies

Researchers have focused on classifying additive relationship word problems since the early 1980s. In particular, two groups of researchers developed similar, but not identical, typologies. The first was developed by Riley, Greeno and Heller (1984). The second was developed by Carpenter, Hiebert and Moser, who were working on their Cognitively Guided Instruction framework. Both groups published a comprehensive version of their typologies (Carpenter & Moser 1983; Riley, Greeno & Heller 1984) but later only included a subset of these problems in their research. Although other research groups have produced typologies, they can all be traced back to these two original typologies.

This section first discusses the category that was 'dropped' by both research groups and explains why it is included in the proposed comprehensive typology used in this article. Secondly, the difference between the two original typologies is highlighted and an argument is made for reflecting this difference in the comprehensive topology. Thirdly, a more recent typology that differs from the original typologies in one important aspect is considered and discarded. These three sections provide the theoretical foundation for the comprehensive and exemplified typology used to analyse the workbooks in this article. The section ends by providing possible reasons for the problematic typologies used in the South African Foundation Phase CAPS. The problems with the typologies in CAPS were described by Roberts (2016a) and the possible reasons for these problematic formulations were uncovered when the early papers about word problem types were studied for this article.

What happened to the 'equalise' problem type?

Carpenter and Moser (1983) and Riley et al. (1984) published comprehensive typologies that included four main problem types. The four categories, the names used by the two research groups and an exemplification of the problem types are detailed in Figure 1.

In this article categories are named according to the labels used by Roberts (2016a). Roberts explains that the label 'change' is preferable to 'join/separate' because it emphasises the common action. 'Collection' is preferable to 'combine' (which implies an action although considered a static problem type) and to 'part–part–whole' (which emphasises a structure shared by all additive relation problem types).

While all work on word problems identifies the difference between dynamic and static problem types, only Carpenter et al. (1981) differentiate between problem types that refer to a single set and those that refer to two separate sets. Figure 2 sets out the relationship between the four problem types and includes the general number sentence for each problem type. While Carpenter et al. (1981) label these different problems as 'set inclusion' and 'no set inclusion' problem types, in this article the labels 'single set' and 'separate sets' are used, as these are considered to be more descriptive. Also note that the names of the categories correspond to the names used in this article.

Used in this article	Riley et al. (1984)	Carpenter and Moser (1983)
Change	**Change** Joe had 3 marbles. Then Tom gave him 5 more marbles. How many marbles does Joe have now?	**Join/Separate** Connie has 5 marbles. Jim gave her 8 more marbles. How many marbles does Connie have altogether?
Equalise	**Equalise** Joe has 3 marbles. Tom has 8 marbles. How many marbles does Joe need to have as many as Tom?	**Equalise** Connie has 13 marbles. Jim has 5 marbles. How many marbles does Jim have to win to have as many marbles as Connie?
Collection	**Combine** Joe has 8 marbles. Tom has 5 marbles. How many marbles do they have altogether?	**Part–part–whole** Connie has 5 red marbles and 8 blue marbles. How many marbles does she have?
Compare	**Compare** Joe has 8 marbles. Tom has 5 marbles. How many marbles does Joe have more than Tom?	**Compare** Connie has 13 marbles. Jim has 5 marbles. How many more marbles does Connie have than Jim?

Source: Adapted from Riley, M., Greeno, J. & Heller, J., 1984, 'Development of children's problem-solving ability in arithmetic', *Development of mathematical thinking,* Academic Press Inc, Orlando, FL. and Carpenter, T.P. & Moser, J.M., 1983, 'The acquisition of addition and subtraction concepts', in R. Lesh & M. Landau (eds.), *Acquisition of mathematics concepts and processes*, pp. 7–44, Academic, New York

FIGURE 1: Comparison of two 'original' word problem typologies.

The category that is most often excluded and that does not appear at all in later work is the 'equalise' problem type. Carpenter and Moser (1983) describe equalise problems as being a combination of change-type problems and compare-type problems (as can also be seen from their position in the grid in Figure 2). Equalise problems are similar to change-type problems because they involve an action (albeit an implicit one) and are similar to compare-type problems because they involve the comparison of two separate sets. It is not clear why equalise problems are no longer referred to in the literature, but I argue that they are an important category when attempting to give a complete range of possible types of word problems.

Is there a difference between 'combine' and 'part–part–whole' problems?

This section discusses the differences between the two 'original' typologies. Even though the two research groups used different names for their problem types, as early as 1982 the categories were mapped onto each other (Nesher, Greeno & Riley 1982). Although the mapping implies that the problem types are the same, a closer look at the exemplification of the categories shows slight differences in one of the categories. To highlight the difference between the categories, the names and numbers from the Riley et al. (1984) typology have been used in both exemplifications (see Figure 3).

One way to understand the difference is in terms of ownership. In Riley et al.'s (1984) example, each of the subsets belongs to a different person. In Carpenter and Moser's (1983) example, the subsets belong to the same person but have different attributes (e.g. colour). This difference is reflected in

	Dynamic (like a movie)	Static (like a photo)
	Change	Collection
Single set	*start ± change = result*	*subset + subset = collection*
	Equalise	Compare
Separate sets	*start ± change = target*	*referent ± difference = compared quantity*

Source: Adapted from Carpenter et al.'s (1981) conceptualisation of the four different types of word problems

FIGURE 2: Four different types of word problems according to Carpenter et al. (1981).

Riley et al. (1984)	Carpenter and Moser (1983)
Combine Joe has 8 marbles. Tom has 5 marbles. How many marbles do they have altogether?	**Part–part–whole** Joe has 8 red marbles and 5 blue marbles. How many marbles does he have?

FIGURE 3: Comparison of 'combine' and 'part-part-whole' problems.

the labels given to the categories by the two research groups. In Riley et al.'s work (1984), where the subsets are owned separately, the problems are known as 'combine' problems – items owned by two individuals need to be combined. In contrast, in Carpenter and Moser's work (1983), the problems are known as part–part–whole problems, thus emphasising the single ownership of the whole set, which has two distinct subsets, identified by different attributes.

Although it is not clear that the distinction between these two exemplifications is sufficient to be evident in learner performance, both types of problems occur in the DBE workbooks and therefore the comprehensive typology developed in this article differentiates between the 'different attributes' and 'different ownership' subcategories for collection-type problems (see Figure 9).

This article argues that the exemplification of the collection-type problems is the main difference between the two original typologies (not the difference in names) and that all other typologies can be traced back to these two classic typologies by considering both the names used for the categories and, more importantly, how the compare-type problems are exemplified. In particular, the extensive work on word problems by De Corte and Verschaffel (1987) uses Riley et al.'s (1984) typology, while the work of Van de Walle, Karp and Bay-Williams (2015) and Clements and Sarama (2009) is based on Carpenter and Moser's (1983) typology.

Is there a difference between the 'larger unknown' and 'referent unknown' compare-type subcategories?

Although the typologies used by Van de Walle et al. (2015) and Clements and Sarama (2009) are based on Carpenter and Moser's typology (1983), they differ in one important aspect. In order to discuss this difference, it is necessary to look at the subcategories used in the different typologies.

Subcategories are created in two ways. The *first* way is by reversing the 'direction' of change in the dynamic problems. This also equates to separating into two the general number sentences (see Figure 2) for the change, equalise and compare problem types.

For example, a change problem where a child is *given* marbles can be changed to a change problem where marbles are *taken away*. A compare problem where the compared quantity is more than the referent can be changed to one where the compared quantity is less than the referent (see Figure 4). Note that such a reversal is not possible for collection-type problems (as can be seen from the general number sentence in Figure 2, which does not include a ±). Instead, the collection-type problems are divided into 'different ownership' and 'different attributes' subcategories. This results in a total of eight subcategories.

The difference between the two change subcategories is made very explicit in Carpenter et al.'s work (1981), which does not include a label for the overarching category but refers to the two subcategories as 'join' and 'separate'. In this article, once again, the labels proposed by Roberts (2016a) – namely 'change increase' and 'change decrease' – are used because this draws attention to 'the problem situation and not the assumed likely operation' (p. 110). By extension this article refers to 'equalise increase' and 'equalise decrease' subcategories.

What is less explicit in the literature is the difference between what is referred to in this article as the 'compare more than' and 'compare fewer than' subcategories. Although these subcategories are exemplified in the original typologies, they are not named. In some other typologies (such as the one used by Roberts [2016a]) only the 'compare more than' problems are exemplified. In the comprehensive typology developed in this article all the possible subcategories are included and named.

The *second* way in which subcategories are created is by changing the position of the unknown in the general number sentence for each of the categories. Figure 5 exemplifies how the change increase category can be used to generate three subcategories by keeping the general number sentence the same (3 + 5 = 8) and changing the position of the unknown.

Note that the collection-type word problems, unlike the other categories, which generate three subcategories, only generate

Change	
Change (increase) ***start + change = result*** Joe had 3 marbles. Then Tom **gave** him 5 more marbles. How many marbles does Joe have now?	**Change (decrease)** ***start - change = result*** Joe has 8 marbles. Then Tom **took** 5 marbles from him. How many marbles does Joe have now?
Compare	
Compare (more than) ***referent + difference = compared quantity*** Joe has 8 marbles. Tom has 3 marbles. How many marbles does Joe have **more than** Tom?	**Compare (fewer than)** ***referent – difference = compared quantity*** Joe has 3 marbles. Tom has 8 marbles. How many marbles does Joe have **fewer than** Tom?

Source: Adapted from Riley, M., Greeno, J. & Heller, J., 1984, 'Development of children's problem-solving ability in arithmetic', *Development of mathematical thinking,* Academic Press Inc, Orlando, FL.

FIGURE 4: Subcategories created by changing direction of action or comparison.

Subcategories	Change (increase) *start + change = result*
Result unknown ***start + change* = [ ]**	Joe had 3 marbles. Then Tom gave him 5 more marbles. How many marbles does Joe have now?
Change unknown ***start* + [ ] = *result***	Joe had 3 marbles. Then Tom gave him some more marbles. Now Joe has 8 marbles. How many marbles did Tom give Joe?
Start unknown **[ ] + *change = result***	Joe had some marbles. Then Tom gave him 5 more marbles. Now Joe has 8 marbles. How many marbles did Joe have in the beginning?

Source: Adapted from Riley, M., Greeno, J. & Heller, J., 1984, 'Development of children's problem-solving ability in arithmetic', *Development of mathematical thinking,* Academic Press Inc, Orlando, FL.

FIGURE 5: Subcategories created by changing the position of the unknown.

two distinct subcategories. This is because there is no need to differentiate between the two subsets, so *collection = subset +* [ ] is the same as *collection* = [ ] + *subset*. This further subdivision results in a total of 22 subcategories as seen in Figure 6.

The labels used for these subcategories are largely based on Roberts (2016a), except for the compare-type problem. For compare problems the label 'compared quantity unknown' (Riley et al. 1984) is used instead of 'whole unknown' because compare-type problems do not have a whole in the sense that collection-type problems do.

A closer look at the typologies used by Van de Walle et al. (2015) and Clements and Sarama (2009) shows an important difference in the way that the compare-type subcategories are generated. Van de Walle et al.'s (2015) typology is discussed here in detail because it is the typology used by most studies that analyse the prevalence of word problems in textbooks (see Despina & Harikleia 2014; Dhindsa et al. 2014; Olkun & Toluk 2006; Parmjit et al. 2006), the one exception being the study by Stigler et al. (1986), which uses the Carpenter and Moser (1983) typology.

The original typologies (Carpenter and Moser [1983] and Riley et al. [1984]) differentiate between the quantity that is being referred to ('the referent'), the quantity that is being compared to the referred quantity ('the compared quantity') and the difference between these two quantities. Van de Walle et al. (2015) and Clements and Sarama (2009), however, differentiate between the larger quantity, the smaller quantity and the difference between the two (see Figure 7). Although both approaches can be used to generate six different subcategories and although the six categories can be mapped onto each other, this article argues that the formulation used in the original typologies is more useful for two reasons.

Firstly, the original typologies foreground the use of the referent in comparison-type problems. As explained by Roberts (2016b), compare type problems are particularly difficult because:

> … the language of comparison is complex in English as comparison requires a referent (something to compare to). The referent is not made explicit ... but is inferred from the word order of the sentence. (p. 5)

	Dynamic (like a movie)	Static (like a photo)
Single set	Change *start ± change = result* *Result unknown* *Change unknown* *Start unknown*	Collection *subset + subset = collection* *Collection unknown* *Subset unknown*
Separate sets	Equalise *start ± change = target* *Target unknown* *Change unknown* *Start unknown*	Compare *referent ± difference = compared quantity* *Compared quantity unknown* *Difference unknown* *Referent unknown*

Source: Adapted from Carpenter, T.P., Hiebert, J. & Moser, J.M., 1981, 'Problem structure and first-grade children's initial solution processes for simple addition and subtraction problems', *Journal for Research in Mathematics Education* 12(1), 27–39. https://doi.org/10.2307/748656.

FIGURE 6: Names of subcategories created by changing the position of the unknown.

For example: *Joe has 5 marbles. Tom has 3 more marbles than Joe. How many marbles does Tom have?* The referent is the number of marbles that Joe has as indicated by 'than Joe'. The referent is not made explicit in Van de Walle et al.'s formulation.

Secondly, in the original typologies the relative difficulty of the compare-type subcategories is linked to the position of the unknown in a similar way to the change-type category. Riley et al. (1984) refer to research that has shown that problems where start set is unknown (for change or equalise problems), or similarly where the referent is known for compare-type problems, are more difficult for children to solve than problems where the start set (change problems) or the referent (compare problems) are known. They explain that the reason for this (p. 177) is that it is more difficult to model problems where the value that needs to be acted on is unknown.

In both the change and the compare problem types, not knowing the value of the set that needs to be acted on increases the difficulty of the problem. In the original typologies, therefore, the more difficult compare-type problems fall into the same subcategories (i.e. referent unknown).

Riley et al. (1984)		
Subcategory	**Compare (more than)** ***referent + difference = compared quantity***	**Compare (less than)** ***referent – difference = compared quantity***
Difference unknown	Joe has 8 marbles. Tom has 3 marbles. How many marbles does Joe have *more than* Tom?	Joe has 8 marbles. Tom has 3 marbles. How many marbles does Tom have *less than* Joe?
Compared quantity unknown	Joe has 3 marbles. Tom has 5 *more* marbles *than* Joe. How many marbles does Tom have?	Joe has 8 marbles. Tom has 5 *less* marbles *than* Joe. How many marbles does Tom have?
Referent unknown	Joe has 8 marbles. He has 5 *more* marbles *than* Tom. How many marbles does Tom have?	Joe has 3 marbles. He has 5 *less* marbles *than* Tom. How many marbles does Tom have?
Van de Walle et al. (2015) (names and numbers changed to match Riley et al.'s 1984 typology)		
Subcategory	**Compare (more than)** ***smaller quantity + difference = larger quantity***	**Compare (less than)** ***larger quantity – difference = smaller quantity***
Difference unknown	Joe has 8 marbles. Tom has 3 marbles. How many *more* marbles does Joe have *than* Tom?	Joe has 8 marbles. Tom has 3 marbles. How many *fewer* marbles does Tom have *than* Joe?
Larger quantity unknown	Tom has 3 marbles. Joe has 5 *more* marbles *than* Tom. How many marbles does Joe have?	Tom has 3 marbles. Tom has 5 *fewer* marbles *than* Joe. How many marbles does Joe have?
Smaller quantity unknown	Joe has 8 marbles. Joe has 5 *more* marbles *than* Tom. How many marbles does Tom have?	Joe has 8 marbles. Tom has 5 *fewer* marbles *than* Joe. How many marbles does Tom have?

Source: Adapted from Riley, M., Greeno, J. & Heller, J., 1984, 'Development of children's problem-solving ability in arithmetic', *Development of mathematical thinking,* Academic Press Inc, Orlando, FL. and Van de Walle, J., Karp, K. & Bay-Williams, J., 2015, *Elementary and middle school mathematics – teaching developmentally*, 9th edn., Pearson, New York.

FIGURE 7: Comparison of Riley's and Van de Walle's compare categories.

For compare-type problems, the difficulty of having the set that needs to be acted on (i.e. the referent) being unknown is compounded by the complexity of English (as explained by Roberts 2016a). While change-type problems indicate that the start is unknown by stating that 'Tom has *some* marbles', the compare-type problems do not make this explicit. Figure 8 shows how to rephrase the compare-type problem to make the unknown referent explicit.

	Compare (more than)	
Subcategory	Referent not explicit	Referent explicit
Referent unknown	Joe has 8 marbles. He has 5 *more* marbles *than* Tom. How many marbles does Tom have?	Tom has some marbles. Joe has 5 more marbles than Tom. Joe has 8 marbles. How many marbles does Tom have?

FIGURE 8: Compare-type problem with unknown referent made explicit.

In contrast, Van de Walle et al. (2015) attribute the relative difficulty of the different subcategories in their typology to the relationship between the quantities and the operations:

> When the larger amount is unknown, stating the problem using the term *more* is easier for children because the relationships between the quantities and the operation more readily correspond to each other. In the smaller unknown situation, stating the problem using the term *fewer* is easier for children for the same reason. (p. 194)

In their typology, therefore, the more difficult problems are not in the same subcategory. This, together with the lack of reference to the referent, is why the formulation of the original typologies is used in this article.

Exemplifying the comprehensive typology

Figure 9 provides a comprehensive typology with all 22 subcategories named and exemplified. The typology is

Action types (movie)

Subcategory	Change	
	Increase	Decrease
Result unknown	Nosisi had 3 marbles. Then she got 5 more marbles. How many marbles does she have now?	Nosis had 8 marbles. Then she lost 5 marbles. How many marbles does she have now?
Change unknown	Nosisi had 3 marbles. Then she got **some** more marbles. Now Nosisi has 8 marbles. How many marbles did she get?	Nosisi had 8 marbles. Then she lost **some** marbles. Now Nosisi has 3 marbles. How many marbles did she lose?
Start unknown	Nosisi had **some** marbles. Then she got 5 more marbles. Now she has 8 marbles. How many marbles did she have at first?	Nosisi had **some** marbles. Then she lost 5 marbles. Now she has 3 marbles. How many marbles did she have at first?

Subcategory	Equalise	
	Increase	Decrease
Change unknown	Nosisi has 3 marbles. Silo has 8 marbles. How many marbles does Nosisi have to win to have the same number of marbles as Silo?	Nosisi has 8 marbles. Silo has 3 marbles. How many marbles does Nosisi have to lose to have the same number of marbles as Silo?
Target unknown	Nosisi has 3 marbles. If she wins 5 marbles she will have the same number of marbles as Silo. How many marbles does Silo have?	Nosisi has 8 marbles. If she loses 5 marbles she will have the same number of marbles as Silo. How many marbles does Silo have?
Start unknown	Silo has 8 marbles. If Nosisi wins 5 marbles she will have the same number of marbles as Silo. How many marbles does Nosisi have?	Silo has 3 marbles. If Nosisi loses 5 marbles she will have the same number of marbles as Silo. How many marbles does Nosisi have?

Static types (photo)

Subcategory	Collection	
	Different attributes	Different ownership
Collection unknown	Nosisi has 3 red marbles and 5 blue marbles. How many marbles does he have altogether?	Nosisi has 3 marbles. Silo has 5 marbles. How many marbles do they have altogether?
Subset unknown	Nosisi has 8 marbles. 3 are red and the rest are blue. How many blue marbles does he have?	Nosisi and Silo have 8 marbles altogether. Nosisi has 3 marbles. How many marbles does Silo have?

Subcategory	Compare	
	More than	Fewer than
Difference unknown	Nosisi has 8 marbles. Silo has 3 marbles. How many marbles does Nosisi have **more than Silo**?	Nosisi has 3 marbles. Silo has 8 marbles. How many marbles does Nosisi have **fewer than Silo**?
Compared quantity unknown	Silo has 3 marbles. Nosisi has 5 more marbles than Silo. How many marbles does Nosisi have?	Silo has 8 marbles. Nosisi has 5 fewer marbles than Silo. How many marbles does Nosisi have?
Referent unknown	Nosisi has 8 marbles. She has 5 more marbles than Silo. How many marbles does Silo have?	Nosisi has 3 marbles. She has 5 fewer marbles than Silo. How many marbles does Silo have?

Source: Author's own work based partly on CAPS document (Department of Basic Education 2011)

FIGURE 9: Exemplification of comprehensive typology.

structured similarly to one of the typologies in the CAPS document (Department of Basic Education 2011:221–222), with the different unknowns in different rows and the different directions of change or comparison in different columns. The names in the problems are also taken from the examples in the CAPS document. For each of the four categories, the 'position of the unknown' subcategories are ordered, with the first being the one learners find easiest to solve (in that category) and the last being the one they find most difficult to solve (Nesher et al. 1982). To emphasise that change-type problems are 'single set' problems, only one person is referred to in the examples, rather than two, which was the case in both original typologies.

What happened to the typologies in the Curriculum and Assessment Policy Standard?

Finally a note on the problematic typology referred to in CAPS in the Grade 2, Term 1 teacher guidelines (Department of Basic Education 2011:221–222). As explained by Roberts (2016a:117), this table is problematic in many ways, most importantly because 'this typology (incorrectly) maps both "join" and "separate" actions and a common "start–change–result" generalised structure onto the static situation of "compare" problem types'.

During the detailed analysis of older texts for the literature review necessary to refine the framework used in this article, a possible reason for these errors was uncovered, namely the potentially ambiguous formatting of tables. The error of mapping 'join' and 'separate' onto categories besides the change category has occurred in at least one other text. This is potentially because of the lack of differentiation in the formatting between the higher-level heading of 'change', and the subheadings of 'join' and 'separate', together with the columns used to list the different examples of word problems. The table on the right of Figure 10 shows the adaptation by Stigler et al. (1986) of the original table (on the left) as it appears in Carpenter and Moser (1983). The headings 'Join' and 'Separate' have been placed above, rather than below, the change heading, therefore implying that they apply to the combine and compare categories as well.

Original typology by Carpenter and Moser (1983:16)

TABLE 2.2

Classification of Word Problems

Join	Separate
Change	
1. Connie had 5 marbles. Jim gave her 8 more marbles. How many marbles does Connie have altogether?	2. Connie had 13 marbles. She gave 5 marbles to Jim. How many marbles does she have left?
3. Connie has 5 marbles. How many more marbles does she need to have 13 marbles altogether?	4. Connie had 13 marbles. She gave some to Jim. Now she has 8 marbles left. How many marbles did Connie give to Jim?
5. Connie had some marbles. Jim gave her 5 more marbles. Now she has 13 marbles. How many marbles did Connie have to start with?	6. Connie had some marbles. She gave 5 to Jim. Now she has 8 marbles left. How many marbles did Connie have to start with?
Combine	
7. Connie has 5 red marbles and 8 blue marbles. How many marbles does she have?	8. Connie has 13 marbles. Five are red and the rest are blue. How many blue marbles does Connie have?
Compare	
9. Connie has 13 marbles. Jim has 5 marbles. How many more marbles does Connie have than Jim?	10. Connie has 13 marbles. Jim has 5 marbles. How many fewer marbles does Jim have than Connie?
11. Jim has 5 marbles. Connie has 8 more than Jim. How many marbles does Connie have?	12. Jim has five marbles. He has 8 fewer marbles than Connie. How many marbles does Connie have?
13. Connie has 13 marbles. She has 5 more marbles than Jim. How many marbles does Jim have?	14. Connie has 13 marbles. Jim has 5 fewer marbles than Connie? How many marbles does Jim have?
Equalize	
15. Connie has 13 marbles. Jim has 5 marbles. How many marbles does Jim have to win to have as many marbles as Connie?	16. Connie has 13 marbles. Jim has 5 marbles. How many marbles does Connie have to lose to have as many marbles as Jim?
17. Jim has 5 marbles. If he wins 8 marbles, he will have the same number of marbles as Connie. How many marbles does Connie have?	18. Jim has five marbles. If Connie loses 8 marbles, she will have the same number of marbles as Jim. How many marbles does Connie have?
19. Connie has 13 marbles. If Jim wins 5 marbles, he will have the same number of marbles as Connie. How many marbles does Jim have?	20. Connie has 13 marbles. If she loses 5 marbles she will have the same number of marbles as Jim. How many marbles does Jim have?

Incorrect formatting by Stigler et al. (1986:158)

TABLE 1

Classification of Word Problems

Join	*Separate*
Change	
1. Connie had 5 marbles. Jim gave her 8 more marbles. How many marbles does Connie have altogether?	2. Connie had 13 marbles. She gave 5 marbles to Jim. How many marbles does she have left?
3. Connie has 5 marbles. How many more marbles does she need to have 13 marbles altogether?	4. Connie had 13 marbles. She gave some to Jim. Now she has 8 marbles left. How many marbles did Connie give to Jim?
5. Connie had some marbles. Jim gave her 5 more marbles. Now she has 13 marbles. How many marbles did Connie have to start with?	6. Connie had some marbles. She gave 5 to Jim. Now she has 8 marbles left. How many marbles did Connie have to start with?
Combine	
7. Connie has 5 red marbles and 8 blue marbles. How many marbles does she have?	8. Connie has 13 marbles. Five are red and the rest are blue. How many blue marbles does Connie have?
Compare	
9. Connie has 13 marbles. Jim has 5 marbles. How many more marbles does Connie have than Jim?	10. Connie has 13 marbles. Jim has 5 marbles. How many fewer marbles does Jim have than Connie?
11. Jim has 5 marbles. Connie has 8 more than Jim. How many marbles does Connie have?	12. Jim has 5 marbles. He has 8 fewer marbles than Connie. How many marbles does Connie have?
13. Connie has 13 marbles. She has 5 more marbles than Jim. How many marbles does Jim have?	14. Connie has 13 marbles. Jim has 5 fewer marbles than Connie. How many marbles does Jim have?
Equalize	
15. Connie has 13 marbles. Jim has 5 marbles. How many marbles does Jim have to win to have as many marbles as Connie?	16. Connie has 13 marbles. Jim has 5 marbles. How many marbles does Connie have to lose to have as many marbles as Jim?
17. Jim has 5 marbles. If he wins 8 marbles, he will have the same number of marbles as Connie. How many marbles does Connie have?	18. Jim has 5 marbles. If Connie loses 8 marbles, she will have the same number of marbles as Jim. How many marbles does Connie have?
19. Connie has 13 marbles. If Jim wins 5 marbles, he will have the same number of marbles as Connie. How many marbles does Jim have?	20. Connie has 13 marbles. If she loses 5 marbles, she will have the same number of marbles as Jim. How many marbles does Jim have?

Source: Carpenter, T.P. & Moser, J.M., 1983, 'The acquisition of addition and subtraction concepts', in R. Lesh & M. Landau (eds.), *Acquisition of mathematics concepts and processes*, pp. 7–44, Academic, New York and Stigler, J.W., Fuson, K.C., Ham, M. & Sook Kim, M., 1986, 'An analysis of addition and subtraction word problems in American and soviet elementary mathematics textbooks', *Cognition and Instruction* 3(3), 153–171

FIGURE 10: Tables showing ambiguous and incorrect formatting for 'Join' and 'Separate' subcategories.

The error of mapping a common 'start–change–result' generalised structure onto the static situation of compare problem types is possibly because of another ambiguously formatted table, this time in Carpenter et al. (1988:388) (see Figure 11). In this table it is not clear whether the headings 'Result unknown', 'Change unknown' and 'Start unknown' refer to only the 'Join' and 'Separate' categories or whether they also refer to combine and compare problem types – although in the text Carpenter et al. (1988) clarify that the headings only refer to the first two categories.

Research methods and design

The comprehensive typology described in the previous section was used to do an analysis of the prevalence and distribution of additive relation word problems in the Grade 1–3 DBE Mathematics workbooks, against a specific typology of word problems.

Selection of texts

The 2017 versions of the DBE foundation phase Mathematics workbooks were selected. The workbooks were selected because in the two schools forming part of the bigger research project, the workbooks were the primary (and in some cases the sole) teaching text used by the foundation phase teachers in 2017.

Procedure

Electronic version of the 12 (four terms per grade across three grades) DBE workbooks were used to identify all additive relation word problems. Firstly, each page was examined for word problems and a copy of the relevant pages was extracted from the electronic document. As in Stigler et al.'s study (1986:160), a problem had to present two premises and a question in order to be included. Word problems that required two steps were excluded (e.g. I bought 15 sweets. I ate 2. I gave my friend 4. How many sweets do I have left?). Almost all of the two-step problems had money (4 out of 8) or weight as a context (3 out of 8). Only one had sweets as a context. Word problems that did not require a numerical answer were also excluded (e.g. I have a R5 and a R1 coin. My friend has three R5 coins. Who has the most money?).

Some problems used pictures of objects to provide some information about the problem, although the question was still stated in words (see Figure 12).

These were taken as word problems. This differs from the decision made by Stigler et al. (1986), who required each premise and each question to be presented in a verbal form. There were only two pages in which this occurred – one in Grade 1 (two questions) and one in Grade 2 (four questions). The one worked example (in the Grade 3 workbooks) was included as one of the word problems.

Each word problem was captured on an Excel sheet and classified according to the three levels of the comprehensive typology. The initial classification was done by the author. The list of word problems together with the typology was then sent to a colleague who independently classified the

Carpenter et al. (1988:388)

Table 1
Classification of Word Problems

Type	Result unknown	Change unknown	Start unknown
Join	1. Connie had 5 marbles. Jim gave her 8 more marbles. How many does Connie have altogether?	2. Connie has 5 marbles. How many more marbles does she need to win to have 13 marbles altogether?	3. Connie had some marbles. Jim gave her 5 more marbles. Now she has 13 marbles. How many marbles did Connie have to start with?
Separate	4. Connie had 13 marbles. She gave 5 marbles to Jim. How many marbles does she have left?	5. Connie had 13 marbles. She gave some to Jim. Now she has 5 marbles left. How many marbles did Connie give to Jim?	6. Connie had some marbles. She gave 5 to Jim. Now she has 8 marbles left. How many marbles did Connie have to start with?
Combine	7. Connie has 5 red marbles and 8 blue marbles. How many marbles does she have?	8. Connie has 13 marbles. Five are red and the rest are blue. How many blue marbles does Connie have?	
Compare	9. Connie has 13 marbles. Jim has 5 marbles. How many more marbles does Connie have than Jim?	10. Jim has 5 marbles. Connie has 8 more than Jim. How many marbles does Connie have?	11. Connie has 13 marbles. She has 5 more than Jim. How many marbles does Jim have?

Grade 2 typology

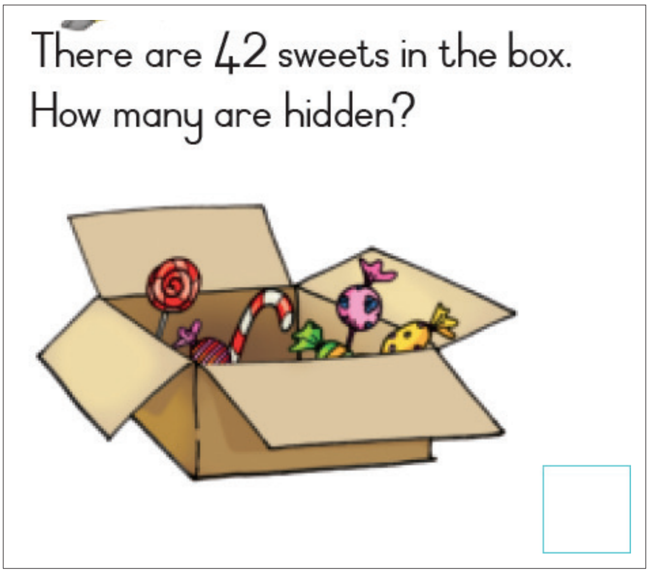
Problem type 1: Change	Join	Separate
Result unknown	Moeketsi has 6 sweets. Mahlodi gives him 9 more. How many sweets does Moeketsi have altogether?	There are 15 sweets. Moeketsie eats 6. How many are left for Mahlodi?
Change unknown	Moeketsi has 6 sweets. How many more does he need to have 15?	Moeketsi has 15 sweets. Mahlodi eats some. There are 9 left. How many did Mahlodi eat?
Start unknown	Moeketsi had some sweets. Mahlodi gives him 9 more. Now he has 15. How many did Moeketsi start with?	Moeketsi eats some sweets. He gave 6 to Mahlodi. Now he has 8 sweets left. How many did he start with?

Problem type 2: Compare	Join	Separate
Result unknown	Moeketsi has 6 sweets.	Mahlodi has 15 sweets.
Change unknown	Mahlodi has 9. How many more sweets does Mahlodi have than Moeketsi?	Mahlodi has 6 sweets. He has 9 fewer sweets than Moeketsi. How many sweets does Mahlodi have?
Start unknown	Mahlodi has 15 sweets. She has 9 more sweets than Moeketsi. How many sweets does Moeketsi have?	Mahlodi has 16 sweets. Moeketsi has 9 fewer sweets than Mahlodi. How many sweets does Mahlodi have?

Problem type 3: Equalise	Join	Separate
Result unknown	Mahlodi has 15 sweets. Moeketsi has 6. How many more sweets must Moeketsi get to have as many as Mahlodi?	Mahlodi has 16 sweets. Moeketsi has 6 sweets. How many more sweets should Mahlodi eat to have the same number as Moeketsi?
Change unknown	Moeketsi has 6 sweets. If he buys 9 sweets he will have as many as Mahlodi. How many does Mahlodi have?	Moeketsi has 6 sweets. If Mahlodi eats 9 sweets she will have the same number of sweets as Moeketsi. How many sweets does Moeketsi have?
Start unknown	Mahlodi has 15 sweets. If Moeketsi buys 9 more sweets he will have the same number of sweets as Mahlodi. How many sweets does Moeketsi have?	Mahlodi has 16 sweets. If she eats 9 sweets she will have the same number of sweets as Moeketsi. How many sweets does Moeketsi have?

Source: Carpenter, T.P., Fennema, E., Peterson, P.L. & Carey, D.A., 1988, 'Teachers' pedagogical content knowledge of students' problem solving in elementary', *Journal for research in Mathematics Education* 19(5), 385–401, viewed 26 February 2018, from http://www.jstor.org/stable/749173 and Department of Basic Education, 2011, *Curriculum and assessment policy standards (CAPS) grades 1–3: Mathematics*, Curriculum document, DBE, Pretoria

FIGURE 11: Tables showing ambiguous and incorrect formatting for 'unknown' subcategories.



Source: Department of Basic Education, 2018, *Workbooks*, viewed 26 February 2018, from https://www.education.gov.za/Curriculum/Workbooks/tabid/574/Default.aspx.

FIGURE 12: Word problem using a picture to provide some information.

Subcategories	Gr 1	Gr 2	Gr 3	Gr 1	Gr 2	Gr 3
	Change					
	Increase			Decrease		
Result unknown	7	4	4	5	7	0
Change unknown	0	0	8	2	0	3
Start unknown	0	0	0	0	0	0
	Equalise					
	Increase			Decrease		
Change unknown	0	0	0	0	0	0
Target unknown	0	0	0	0	0	0
Referent unknown	0	0	0	0	0	0
	Collection					
	Attributes			Ownership		
Collection unknown	1	0	2	6	3	1
Subset unknown	1	4	0	0	0	0
	Compare					
	More than			Fewer than		
Difference unknown	1	0	1	0	0	0
Compared quantity unknown	0	0	0	0	0	1
Referent unknown	0	0	0	0	0	0

Gr, grade.

FIGURE 13: Frequency of different types of word problems, by grade.

61 problems. Any cases where there were differences were discussed until an agreement was reached.

Results

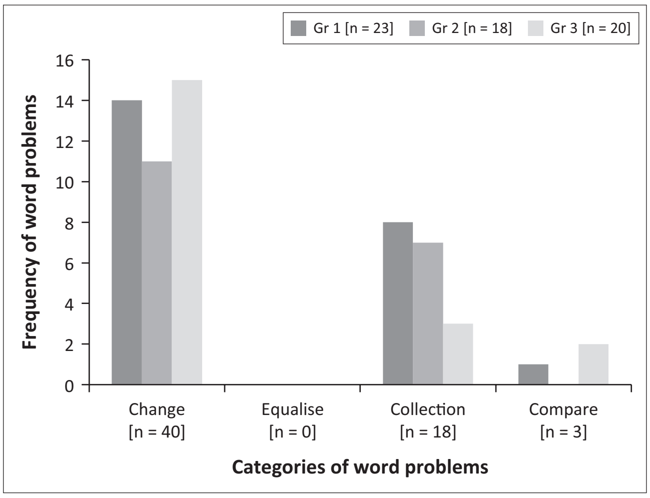
The frequency of each of the 22 problem types in the comprehensive typology, by grade, is shown in Figure 13. The first interesting thing to note is the very small total number of word problems (61) that appear in the workbooks across the three grades.

Looking at the distribution of word problems across the four main problem types reveals a greater focus on change-type problems, no equalise-type problems and very few compare-type problems. In fact, only three compare-type word problems appear across all three grades:

- Thandi has 19 bananas. Silo has 10 bananas. How many *more* bananas does Thandi have *than* Silo?
- Sipho counts out 87 bottle tops. Andile counts out 38. How many *more* bottle tops does Sipho count *than* Andile?
- The Grade 2s have a collection of 360 marbles. The Grade 3s have 216 *fewer* marbles *than* the Grade 2s. How many marbles do the Grade 3s have?

It should be noted that there were three additional compare-type problems that were not included in the analysis because they did not require a numerical answer. However, Roberts (2016b) argues that these types of problems are an important step in supporting learners to solve compare type problems. The problems provide two amounts (e.g. Mandla has 5 pencil crayons. Anne has 8 pencil crayons) but instead of asking 'How many fewer pencil crayons does Mandla have than Anne?', learners are asked 'Who has fewer pencil crayons?'. This type of question can only be asked for compare-type problems and therefore there are no similar occurrences for the other problem types.

In terms of the distribution across the grades, the total number of word problems varies across the grades from 23 in Grade 1, to



Gr, grade.

FIGURE 14: Distribution of word problems across four main categories by grade.

18 in Grade 2 to 20 in Grade 3, as seen in Figure 14. The number of word problems in each category also varies across the grades, with Grades 1 and 3 having more change and compare-type problems than Grade 2 and the number of collection-type problems decreasing with each grade (see Figure 14).

In terms of the distribution across the subcategories, there are a number of interesting things to note. For the change-type problems, there are more increase than decrease-type problems. Result unknown problems are more prevalent than change unknown problems and there are no start unknown problems. Similarly, there are no compare-type problems where the set to be acted on (i.e. the referent) is unknown.

For collection-type problems, both subcategories (different attributes and different ownership) appear in the workbooks. However, all the 'different ownership' problems are in the 'subset unknown' category while the 'different attributes' problems are almost equally spread between the 'collection unknown' and 'subset unknown' subcategories (see Figure 15).

Subcategories	Change	
	Increase	Decrease
Result unknown	15	12
Change unknown	8	5
Start unknown	0	0
	Equalise	
	Increase	Decrease
Change unknown	0	0
Target unknown	0	0
Referent unknown	0	0
	Collection	
	Attributes	Ownership
Collection unknown	3	10
Subset unknown	5	0
	Compare	
	More than	Fewer than
Difference unknown	2	0
Compared quantity unknown	0	1
Referent unknown	0	0

FIGURE 15: Frequency of different types of word problems across three grades.

Discussion

In this section each of the four main results are discussed and related recommendations are made. The strengths and limitations of the research are also considered.

Firstly, the analysis of the foundation phase workbooks showed a very small number of additive relation word problems across 3 years of schooling in comparison to textbooks used in other countries. For example, the US Grade 1–3 textbooks analysed by Stigler et al. (1986) have a total of between 347 and 408 word problems while the Soviet textbooks have a total of 275 word problems. Turkish Grade 1–3 textbooks have between 139 (Tarim 2017) and approximately 190 (Olkun & Toluk 2006) word problems, while one set of Brunein textbooks was shown to have 389 word problems (Dhindsa et al. 2014). In comparison, the DBE workbooks only included a total of 61 word problems.

This considerably smaller number of word problems is not problematic if the workbooks are being used as they were designed to be used – namely as supplementary resource for learners to practise what they have been taught in class. However, when they are being used as the sole teaching resource and learners not being exposed to any other word problems, the very small number of problems is a matter for concern. Should the workbooks be revised, an increase in the number of word problems across the three grades is recommended.

Secondly, the distribution of word problems across the four main problem types reveals a similar distribution to other studies – namely a greater focus on change-type problems, no equalise-type problems and very few compare-type problems (Despina & Harikleia 2014; Dhindsa et al. 2014; Olkun & Toluk 2006; Singh 2006; Stigler et al. 1986; Tarim 2017). The complete lack of equalise problems is particularly striking. The only study to classify equalise-type problems was by Stigler et al. (1986), who included only one subcategory of equalise problems (equalise increase, change unknown). Only two of the four textbooks that were analysed in their study had this type of equalise problem. All other studies used Van de Walle et al.'s (2015) typology, which does not include the equalise problem-type.

As argued previously, equalise problems can be used to support learners to move from change-type problems (which they generally find easier to solve), through equalise problems, to the more difficult compare-type problems. The complete absence of equalise problems in the DBE workbooks and in most other studies points to the likelihood that even in classes where the workbooks are not the primary teaching text, learners are not being presented with equalise problems.

The under-representation of collection- and compare-type problems (18 and 3, respectively) is a reason for concern as these two-word problem types provide learners with an opportunity to engage with useful mathematical concepts. Collection-type problems highlight that a set can be separated into subsets. Compare-type problems introduce learners to an alternative conceptualisation of subtraction – as the difference between two quantities rather than as a 'take away sum'. In more advanced Mathematics the notion of the difference is much more prevalent than that of subtraction as 'taking away'.

Thirdly, in terms of the distribution across the grades, the numbers of problems in each grade (23 in Grade 1; 18 in Grade 2; and 20 in Grade 3) are too small and too similar to draw any reasonable conclusions. They do, however, emphasise the very small number of problems learners are presented with in the workbooks, particularly in Grade 2, which does not include any compare-type problems. A more preferable distribution would include more problems in each grade, some equalise problems (particularly in Grades 1 and 2) and a decrease in change-type problems, mirrored by an increase in compare-type problems (which are conceptually more challenging for learners) across the three grades.

Finally, the distribution of word problems across the subcategories shows an under-representation of the subcategories that learners find more difficult to solve. Nesher et al. (1982) reported on the results of a number of studies that determined the relative difficulty of different word problems. These studies showed that, for change-type problems, children find 'result unknown' problems easier to solve than 'change unknown' problems, and they find 'start unknown' problems the most difficult to solve. For collection-type problems, the 'collection unknown' subcategory is significantly easier than 'subset unknown' subcategory. For compare-type problems, 'referent unknown' problems are more difficult than 'compared quantity unknown' or 'difference unknown' problems.

This increase in difficulty is reflected in the comprehensive typology where the easiest subcategory appears first for each problem type and the most difficult one last. In each category the position of the unknown determines the relative difficulty of the subcategory, as can be seen in Figure 5, where the general number sentence for each of most difficult (last) subcategories has the unknown in the first position. Figure 15 shows a decrease in the number of problems, as the difficulty level of the problems increase (darker cells equate to better-represented subcategories). Therefore, the problem types that learners find more difficult to solve are the ones that are less prevalent in the workbooks. This is another reason for concern and a more preferable distribution would be one in which the number of problems in the more difficult subcategories increased with an increase in grade.

Conclusion

This article sets up a comprehensive typology for classifying additive relation word problems and then uses the typology as an analytical framework to explore the distribution of word problems in the DBE foundation phase Mathematics workbooks. The comprehensive typology reintroduces equalise-type problems, which have been excluded from more recent typologies and argues that these are a useful bridge from change-type problems to the more difficult compare-type problems.

The comprehensive typology also distinguishes between two subcategories that were previously grouped together, namely the part–part–whole category (from the work of Carpenter and Moser) and the combine category (from the work of Riley and Greeno). These two subcategories have been renamed 'different attributes' and 'different ownership'. A more recent typology by Van de Walle, which differs from the early typologies in terms of the collection problem type, was considered but not used as the formulation of the subcategories in the compare category does not take into consideration the importance of the referent. The article also provides possible reasons for the problematic typologies that are presented in the CAPS document. The layout and choice of contexts and names in the comprehensive typology is aligned with the typologies in the CAPS document.

The analysis of the DBE foundation phase Mathematics workbook reveals a very limited number of word problems (only 61 across three grades). In particular, there are no equalise problems and only three compare-type problems. This is a reason for concern in the light of the central role played by the DBE workbooks in many no-fee schools. The very limited number of compare-type word problems is particularly concerning because of the consistently poor performance on word problems as highlighted in the 2012, 2013 and 2014 ANA diagnostic reports.

In their analysis of the Grade 3 DBE workbooks as curriculum tools, Hoadley and Galant (2016) conclude that it is not viable to use the current workbooks as a teaching or transmission tool, primarily because of the limitations of conceptual signalling. The findings of this article support this conclusion by showing that the frequency and distribution of the additive relation word problems in the workbooks are not suitable for them to be used as a teaching tool.

Acknowledgements

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Competing interests

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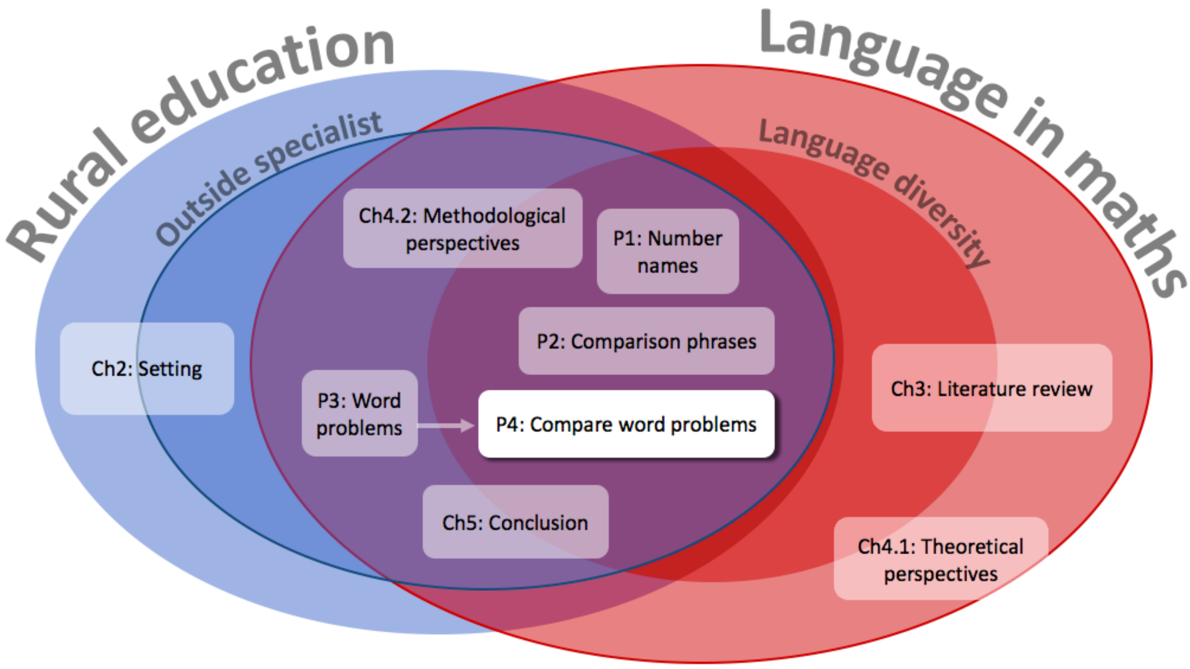
PAPER 4: Relative difficulty of early grade comparison word problems: Learning from the case of isiXhosa



The fourth paper at the core of this study considers the same central component of early grade mathematics as the second paper, namely, comparison. However, unlike the second paper, the fourth paper considers comparison questions rather than comparison phrases. In particular, the fourth paper considers the relative difficulty of comparison word problems and the factors that contribute to their relative difficulty. Because the paper considers both isiXhosa and English comparison questions, and because the examples of comparison questions are taken from canonical texts, this paper (like the first and second papers) falls within the intersection of language diversity and rural education, as undertaken by an outsider. The unit of analysis of the fourth paper is the collection of comparison questions in the South African version of the EGMA as well as the comparison questions in the CAPS and the DBE, NECT and NMI workbooks.

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Relative difficulty of early grade compare type word problems: Learning from the case of isiXhosa

Author:
Ingrid E. Mostert[1]

Affiliation:
[1]Centre for Education Practice Research, Faculty of Education, University of Johannesburg, Soweto, South Africa

Corresponding author:
Ingrid Mostert,
ingridmostert@gmail.com

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Word problems form an important part of the early grade mathematics curriculum in South Africa. Studies have shown that the relative difficulty of word problems differ: learners are more likely to solve certain types of word problems than others, with compare type problems being the most difficult. In order to help early grade learners understand and solve compare problems, it is important to understand the relative difficulty of different types of compare type problems and the factors that contribute to their relative difficulty. While these factors have been studied in English, less research has attended to word problems in other languages, such as isiXhosa. In this study a typology of isiXhosa compare type (difference unknown) word problem was set up. The typology included two dimensions, namely the problem situation and the comparative question. The relative difficulties of specific word problems from this typology were compared by analysing the results from an early grade mathematics assessment administered to two cohorts of Grade 1–3 isiXhosa learners in five rural Eastern Cape schools. The analysis showed that in isiXhosa, as in English, some compare type problems are easier to solve than others. Problems with 'matching' situations are easier to solve than problems with 'no matching' situations. Problems with alternatively formulated comparative questions, specifically those using *-shota* or *kangakanani*, are easier to solve than those using a more classic formulation. This study highlights the importance of understanding the ways in which African languages express mathematical ideas in order to identify and leverage affordances for teaching and learning mathematics.

Keywords: word problems; early grade mathematics; compare problems; isiXhosa; mathematical language.

Introduction

Word problems are a central, yet hard-to-teach, aspect of early grade mathematics. For example, in South Africa word problems have been identified as a recurring weakness in the South African Annual National Assessments (ANAs) (Department of Basic Education, 2012, 2014, 2015). Research has shown that the relative difficulty of word problems differs: learners are more likely to solve certain types of word problems than others. For additive relation word problems, in other words any word problems involving addition and subtraction, compare type problems have been shown to be the most difficult for learners to solve. Compare type problems are of the form 'Sbu has eight bananas and Sive has five bananas. How many more bananas does Sbu have than Sive?' While there has been some research into early grade word problems in South Africa (e.g. Petersen, McAuliffe, & Vermeulen, 2017), and some research into word problems and African languages in higher grades (e.g. Sepeng, 2013), there has been little research into early grade word problems in African languages. This is problematic as more than 75% of learners are taught mathematics in an indigenous African language in the first four years of formal schooling (Spaull, 2016).

Internationally, different types of additive relation word problems and their relative difficulty (as measured by the percentage of learners who correctly answered the problem out of the total number of learners who were asked the question) have been studied in relation to English since the late 1970s. This work was pioneered by two research groups: the first led by Carpenter, Hiebert and Moser (1981) and the second by Riley, Greeno and Heller (1983). Such studies have shown that there are a number of factors that influence the relative difficulty of different word problems. These include general factors such as problem length, grammatical complexity and whether learners use concrete aids or not, as well as specific factors such as semantic structure and the position of the unknown (Riley et al., 1983). Many of these factors relate to language. This raises questions regarding the extent to which these factors influence the relative difficulty of word problems in languages other than English, especially languages with linguistic features significantly different to English, such isiXhosa, one of South Africa's nine official African languages.

This study contributes to the limited research on early grade word problems in languages other than English by examining compare type word problems in isiXhosa. The study offers a sub-typology for compare type (difference unknown) problems, both in English and in isiXhosa. The relative difficulty of the problems in the isiXhosa sub-typology is then empirically tested using data collected from an adapted early grade mathematics assessment (EGMA).

The following two research questions are answered in this study:

- Are certain types of isiXhosa compare type problem easier for learners to solve than others?
- If so, is the relative difficulty influenced by:
  - the formulation of the comparative question
  - the problem situation?

Theoretical and methodological perspectives

This article is informed by theoretical perspectives and methodological tools from linguistics that have proved helpful for research into the way different languages express mathematical concepts, as proposed in a recent paper by Edmonds-Wathen (2019). This article also draws on theoretical perspectives from variation theory, a general theory of learning largely developed by Marton and Booth (1997) and later extended by Watson and Mason (2005) in relation to mathematics learning.

Edmonds-Wathen (2019) proposes using a typological framing for research on the diversity of mathematical expression in different languages and using interlinear morphemic glossing to present examples in different languages. These perspectives and methodologies are particularly pertinent for studies done by a researcher not fluent in the language that is being studied. Edmonds-Wathen points out that linguists often work with languages that they are unfamiliar with, either by working with translated texts or by working closely with bilingual speakers. She argues that mathematics education researchers can, and do, work in similar ways, with this study being a case in point. This study was undertaken by an English speaker with an emergent understanding of isiXhosa. The researcher worked very closely with a number of isiXhosa speakers to deepen her understanding of isiXhosa, particularly in relation to compare type problems.

Typological framing

Typology is an area of linguistics that describes and classifies languages according to their structural similarities and differences (Edmonds-Wathen, 2019). Typology strives to compare languages through an analysis that is framework-neutral (Nichols, 2007). Edmonds-Wathen (2019) argues that because of this neutrality, 'a typological approach may be useful to investigate mathematical expression in different languages, without privileging one language over another' (p. 121). This is particularly important when comparing a language with a well-developed mathematical register (see Halliday, 1978), for example English, with a language without a formal mathematical register or with a mathematical register that is still being strengthened, for example isiXhosa.

While this study does not adopt a strict typological approach, the study does strive to ensure that English was not privileged over isiXhosa, for example by ensuring that a range of ways of expressing comparative questions in isiXhosa was studied and not only those that correspond to the way in which English expresses comparative questions. However, as the researcher is not an isiXhosa speaker, English and the linguistic features used to express comparative questions in English provided a starting point for the study, therefore implicitly privileging English.

Interlinear morphemic glossing

One of the challenges of researching how different languages express mathematical ideas is how to present examples from a language different to the language of publication. To overcome this challenge, Edmonds-Wathen (2019) suggests using a simplified interlinear morphemic gloss.[1] Interlinear morphemic glossing allows the structure of the example to be presented. Often the structure is lost if only an idiomatic translation is provided (Edmonds-Wathen, 2019). Interlinear morphemic glossing is particularly helpful when presenting data from languages where word order is not necessarily a determinant of the function of a word.

An interlinear morphemic gloss consists of four levels (Edmonds-Wathen, 2019). For this article the top level gives the isiXhosa in sentence form. The second level gives the isiXhosa morphemes (the smallest unit of a language that has its own meaning), the third level gives the English morphemic gloss, and the final level gives a free translation in English. Morphemes are separated by a hyphen. If a morpheme is translated by more than one word the words are separated by a full stop. For example:

Level 1: Umama uneembiza ezisibhozo neziciko ezihlanu.

Level 2: *umama u-nee-mbiza ezi-sibhozo ne-ziciko ezi-hlanu*

Level 3: mother she.is-with-pots that.are-eight with-lids that.are-five

Level 4: Mother has eight pots and five lids.

The simplified interlinear morphemic gloss is used in order to make the examples accessible to a mathematics education audience. For this reason, in some instances in this article, not every morpheme is glossed separately.[2] For example, nouns and their prefixes, which indicate the noun class and the

1. See Comrie, Haspelmath and Bickel (2008) for the Leipzig glossing rules used widely in linguistics.

2. IsiXhosa is an agglutinative language and words are made up of many morphemes.

number (singular or plural) of the noun, are not glossed separately. In cases where strict glossing would detract from the comparisons being made, other glossing rules have also not been strictly followed.

Language of variation

Variation theory is a general theory of learning that has been applied specifically to learning mathematics. A central notion within variation theory is that in order to discern one aspect of a phenomenon, that aspect needs to be varied while other aspects remain unchanged (Al-Murani, Kilhamn, Morgan, & Watson, 2019). The aspect of the phenomenon that varies is called the 'dimension of variation'.

Watson and Mason (2005) extended these ideas by defining the variation that is possible within a 'dimension of variation' as the 'range of change'. For example, in the expression $x + 3$, one of the dimensions of variation is the addend (others include: the letter representing the variable, the operator and the order in which the variable and constant appear in the expression). The values that the addend can take (i.e. natural numbers, negative numbers, rational numbers and so on) constitute the range of change of this particular dimension of variation (Al-Murani, 2006). The extent to which a learner can discern the dimensions of variation and the corresponding 'range of change' of the expression $x + 3$ is an indication of how well the learner understands the algebraic expressions.

In this article the ideas of a 'dimension of variation' and of a 'range of change' are applied not to an object of learning, but to an object of study, namely compare type word problems. In order to explore the full range of possible compare type problems, different dimensions of variation were identified and varied to set up a typology of compare type problems for isiXhosa and for English.

Additive relation word problems

Additive relation word problems and the factors that influence their relative difficulty have been studied for English word problems since the late 1970s. In the following two sections relevant studies are discussed.

Word problem typologies

Early researchers categorised word problems describing the same mathematical problem but using different semantic structures into typologies of word problems (e.g. Carpenter & Moser, 1983; Riley et al., 1983). Recently these typologies have been combined into one comprehensive typology (Mostert, 2019). The categories and labels from this comprehensive typology will be used in this article. At the highest level this typology consists of four different types of word problems, differing in terms of the number of sets being compared and whether the problem is dynamic or static (Figure 1).

Each of these four main categories of word problems can be separated into subcategories in two ways. Firstly, each category can be separated into two subcategories based on a number of factors or dimensions: the 'direction' of the change or equalisation, whether attributes or ownership are different in collection problems and whether the comparison is 'more than' or 'less than'. Secondly, by changing the position of the unknown, each category can further be divided into two or three subcategories resulting in a total of 22 subcategories (see Figure 2).

The word problems studied in this article are 'more than' compare problems where the difference is unknown (marked with † in Figure 2).

Factors influencing difficulty of word problems

As mentioned previously, there are a number of different factors that have been identified as influencing the difficulty level of word problems. The factor that is relevant for this study is the clarity of the problem for learners, both in terms of the problem situation and in terms of the formulation of the comparative question. Verschaffel and De Corte (1993) report that most mistakes made by young children when solving additive relation word problems are more likely to be because they represent the problem situation incorrectly, not, as was formerly widely believed, because they choose the incorrect arithmetic operation. This is evident from a number of studies, referred to and validated by Verschaffel and De Corte (1993), which demonstrate that problems can be rephrased, without

	Dynamic (like a movie)	Static (like a photo)
Single set	**Change**	**Collection**
	A girl has five sweets. Then she gets two more sweets. How many sweets does she have now?	A girl has five red sweets and two blue sweets. How many sweets does she have altogether?
Separate sets	**Equalise**	**Compare**
	A girl has seven sweets. A boy has five sweets. How many more sweets does the boy need to have the same number of sweets as the girl?	A girl has seven sweets. A boy has five sweets. How many more sweets does the girl have than the boy?

Source: Adapted from Mostert, I. (2019). Distribution of additive relation word problems in South African early grade Mathematics workbooks. *South African Journal of Childhood Education*, *9*(1), 1–12. https://doi.org/10.4102/sajce.v9i1.655

FIGURE 1: Four main categories of word problems.

Change		Collection	
Increase	*Decrease*	*Attributes*	*Ownership*
Result unknown	Result unknown	Collection unknown	Collection unknown
Change unknown	Change unknown	Subset unknown	Subset unknown
Start unknown	Start unknown		
Equalise		**Compare**	
Increase	*Decrease*	*More than*	*Less than*
Change unknown	Change unknown	Difference unknown†	Difference unknown
Target unknown	Target unknown	Compared quantity unknown	Compared quantity unknown
Start unknown	Start unknown	Referent unknown	Referent unknown

Source: Summarised from Mostert, I. (2019). Distribution of additive relation word problems in South African early grade Mathematics workbooks. *South African Journal of Childhood Education*, *9*(1), 1–12. https://doi.org/10.4102/sajce.v9i1.655

†, The word problems studied in this article are 'more than' compare problems where the difference is unknown.

FIGURE 2: Twenty-two subcategories in the comprehensive typology.

changing their semantic structure, in way that is easier for learners to correctly represent the problem situation.

For collection problems there are two cases of rephrasing that have been shown to increase the likelihood of learners solving the problem correctly. Firstly, Carpenter et al. (1981) found that if the problem 'There are six children on the playground. Four are boys. How many are girls?', was changed to 'There are six children on the playground. Four are boys and *the rest are girls*. How many are girls?', a higher percentage of learners answered the question correctly.

Secondly, Lindvall and Ibarra (1980) found that when the problem 'Together Tom and Joe have eight apples. Three apples belong to Tom. How many belong to Joe?' was changed to 'Together Tom and Joe have eight apples. Three *of these* apples belong to Tom. How many *of these* belong to Joe?', the problem was significantly easier for kindergarten children to solve correctly.

Such rephrasing is also possible for compare type problems, with some empirical data showing that rephrasing can increase the percentage of learners who correctly solve the problem. This will be discussed in detail in the next section.

Compare type problems in English

Compare type problems have been identified as the most difficult type of additive relation word problem for young children to solve (Fuson, Carroll, & Landis, 1996). At least part of the reason why learners struggle to solve compare type problems is because standard compare type problems ('Sbu has eight bananas and Sive has five bananas. How many more bananas does Sbu have than Sive?') include quantifiers such as 'more than' and 'less than'. Quantifiers form part of later-developing languages skills, skills that mother tongue speakers continue to develop up to approximately age 9 (Berman, 2004). If children are still learning to understand and use words such as 'more than' and 'less than' it is not surprising that they struggle to represent the correct problem situation for compare type problems.

Another reason why compare problems are so difficult for young learners to solve is that learners confuse the 'classic' comparative question 'How many more?' with the question 'How many?' or with the question 'Who has more?'. For example, the problem 'Sbu has eight bananas and Sive has five bananas. How many more bananas does Sbu have than Sive?', is often answered with 'eight' (correctly answering the question 'How many bananas does Sbu have?') or with 'Sbu' (correctly answering the question 'Who has more bananas?') (Roberts, 2016). In response to this potential confusion, Roberts (2016) suggests first asking 'Who has more bananas?', then 'How many bananas does Sbu have?', before asking 'How many more bananas does Sbu have than Sive?'.

In terms of comparison, both mathematics education and linguistic research (e.g. Kennedy, 2009) has focused on 'more than' compare type problems, neglecting 'less than' compare type problems (see Figure 2). This focus on 'more than' problems in the literature is reflected in assessments, such as the EGMA, which only includes 'more than' compare type problems. Because this study analyses data from the EGMA, only 'more than' compare type problems are considered. This is a limitation of the study and of research in general as 'more than' compare type problems are not necessarily equivalent to 'less than' compare type problems, especially for non-Indo-European languages. IsiXhosa is one such language where they are not equivalent.

It is also important to note that while there are three subcategories of compare type (more than) word problems, as can be seen from Figure 2 and as exemplified in Table 1, this study only considers 'difference unknown' compare type problems.

TABLE 1: Exemplification of subcategories of compare (more than) problems.

Subcategory	Example
Difference unknown	A boy has nine sweets. A girl has eleven sweets. How many more sweets does the girl have than the boy?†
Compared quantity unknown	A boy has nine sweets. A girl has two more sweets than the boy. How many sweets does the girl have?
Referent unknown	A boy has some sweets. A girl has two more sweets than the boy. The girl has eleven sweets. How many sweets does the boy have?

Source: Extracted from Mostert, I. (2019). Distribution of additive relation word problems in South African early grade Mathematics workbooks. *South African Journal of Childhood Education*, *9*(1), 1–12. https://doi.org/10.4102/sajce.v9i1.655

†, The word problems studied in this article are 'more than' compare problems where the difference is unknown.

Different types of compare type problems

In this section, three variations of the standard, 'difference unknown' compare type problem are discussed.

In as early as 1980, Hudson constructed and tested a variation of the standard compare type problem. Hudson's (1980) variation differed from the standard compare type problem in two ways. Firstly, the problem situation was set up to invoke the idea of matching by choosing birds and worms as the subjects of the story, namely 'There are five birds and four worms'. Secondly, the 'classic' phrasing of the comparative question 'How many more birds than worms are there?', was rephrased as 'Suppose the birds all race over and each one tries to get a worm! Will every bird get a worm? How many birds won't get a worm?'.

Hudson (1980) then presented learners with the same problem situation about the five birds and four worms, but posed the comparative question in two different ways, using the classic 'how many more' formulation and using the 'how many won't get' formulation. Table 2 shows the striking difference in results with a much higher percentage of learners being able to solve the problem with the 'how many won't get' formulation.

A second variation was introduced by Roberts (2016), in what she refers to as 'compare (matching) problems'. These are compare type problems that draw attention to the absence of elements by asking 'How many elements are missing?'. An example of Roberts's compare (matching) problem is: 'There are 11 locks but only 9 keys. How many keys are missing?'. Roberts explains that:

> the choice of locks and keys is deliberate, as in this problem context it is implicit that each key fits uniquely with a particular lock. This unique 1:1 matching of each element in one set to each element in another set is not explicitly implied in the compare problem 'I have 11. You have 9. How many more do you have than me?' (p. 68)

The unique one-to-one matching of Roberts's compare (matching) problem is *embedded* in the problem situation. As Roberts (2016) points out, this is not the case for 'standard' compare type problems. It is also not the case for Hudson's 'won't get' problems, where, unlike with locks and keys, one bird can get two worms or two birds share one worm. In Hudson's (1980) variation, the one-to-one matching is *imposed* on the problem situation by adding the phrase 'each [*bird*] tries to get a worm'.

TABLE 2: Percentage of children with consistent correct responses.

Grade	How many more? (%)	How many won't get? (%)
Nursery school	17	83
Kindergarten	25	96
First grade	64	100

Source: Reproduced from Riley, M., Greeno, J., & Heller, J. (1983). Development of children's problem-solving ability in arithmetic. In H.P. Ginsburg (Ed.), *The development of Mathematical thinking* (pp. 153–196). New York, NY: Academic Press.

TABLE 3: Percentage of children with consistent correct responses.

Assessment	How many more?† (%)	How many missing?‡ (%)
Cycle 2 pretest	52	68
Cycle 3 pretest	34	73

Source: Collated from Roberts, N. (2016). *Telling and illustrating additive relations stories: A classroom-based design experiment on young children's use of narrative in mathematics*. Johannesburg: University of Witwatersrand. https://doi.org/10.13140/RG.2.1.1092.6964

†, Referred to as Compare (disjoint set) by Roberts (2016).

‡, Referred to as Compare (matching) by Roberts (2016).

Type	Example
Standard	A girl has seven sweets. A boy has five sweets. How many more sweets does the girl have than the boy?
Roberts (2016)	A mother has seven pots and five lids. How many lids are missing?
Early grade mathematics assessment (EGMA)	A mother has seven children and five oranges. How many oranges are still needed if the mother wants to give each child one orange?
Hudson (1980)	There are five birds and four worms. Suppose the birds all race over and each one tries to get a worm! How many birds won't get a worm?

Note: Question words are underlined, and phrases that impose one-to-one matching are highlighted.

FIGURE 3: Different types of compare type problems in the literature.

While Roberts (2016) did not use the same problem situation, she did empirically determine the facility score of a compare (matching) problem and a standard compare type problem (which she refers to as a compare (disjoint set) problem). For both pretests in her study the facility score of the compare (matching) problem was much higher than that of the standard compare type problem (see Table 3).

A third variation of the standard compare type problem appears in the EGMA used in this study (Figure 3). This variation is similar to Hudson's 'won't get' variation. Firstly, the problem situation is set up to invoke matching, this time between children and oranges: 'A mother has seven children, and she has two oranges'. Secondly, the need for one-to-one matching is *imposed* on the situation rather than embedded in the situation. This is done through the phrase 'if the mother wants to give each child one orange'. However the EGMA variation differs from the Hudson (1980) variation in that rather than asking a 'won't get' question, a 'still needed' question is asked: 'How many oranges are still needed?'.

While the three variations of the standard compare problem, summarised in Figure 3, are helpful in showing that rephrasing a question can influence how easy or difficult it is for learners to solve, the problems differ both in terms of the problem situation and in terms of the formulation of the comparative question. This means that it is not possible to isolate the effect of the different factors on the level of difficulty of the different problems. In the next section a typology of compare type problems is set up, taking into account the variation that is possible for both factors.

Typology of English compare type problems

Drawing on the language of variation theory, the 'dimensions of variation' for compare type problems are (1) the problem situation and (2) the formulation of comparative question. The problem situation can either be one that invokes matching by referring to two things that learners might expect to go together (e.g. locks and keys or children and oranges) or one that does not invoke matching by referring to things that do not necessarily go together (e.g. sweets belonging to a girl and sweets belonging to a boy). For matching problems, it possible to further differentiate between problems that have one-to-one matching embedded in the situation and those in which the one-to-one matching is not embedded.

In English, the comparative question, which constitutes the second dimension, can either be formulated in the 'classic' form, 'how many more?' or, for matching situations, the question can be formulated in one of a number of alternative ways such as 'how many are missing?' or 'how many are still needed?'.

Using these two dimensions of variation and the range of change that is permissible for each dimension, it is possible to set up a typology of compare type problems. Figure 4 provides an overview of the typology as well as showing how each of the four compare type problems discussed previously (see Figure 3) fits into the typology. Appendix 1, Figure 1-A1 exemplifies each of the categories in the typology.

There are few important things to note about the typology.

Firstly, while it is possible to ask a classically formulated comparative question with a matching problem situation (e.g. 'A mother has eight pots and five lids. How many more pots are there than lids?'), it is not possible to use an alternatively phrased question with a 'no matching' problem situation – the problem 'A girl has seven sweets. A boy has five sweets. How many sweets are missing?' does not make sense. This is the reason for the *n/a* cell.

Secondly, when a problem has a matching problem situation where one-to-one matching is not embedded but an alternatively phrased comparative question is used, an additional phrase (such as 'each bird tries to get a worm') must be added in order to impose the one-to-one matching on the situation. For this reason, the typology differentiates between '1-to-1 matching not embedded' and '1-to-1 matching imposed'. See Appendix 1, Figure 1-A1 for examples of word problems in the different categories. Finally, it is important to remember that this typology is only for 'difference unknown' compare type problems.

Once a broader typology has been set up showing the dimensions along which the problems can vary, it is possible to compare problems that only vary in terms of one dimension (either the problem situation or the comparative question) in order to establish the extent to which each factor influences the relative difficulty of compare type problems. In this study the influence of these two dimensions is explored for isiXhosa compare type problems. In order to do this a typology for isiXhosa compare type problems is set up, drawing on examples from canonical texts.

Compare type problems in isiXhosa

In a previous study, Mostert and Roberts (2020) describe the linguistic features of comparative phrases in isiXhosa. In order to do this they analysed the examples of comparative phrases appearing in four canonical texts,[3] written in English and translated into isiXhosa (Mostert & Roberts, 2020). This set of examples included both comparison phrases (e.g. 'There are more dogs than cats') as well as comparative questions (e.g. 'How many more dogs are there than cats?'). While the previous study only focused on the comparison phrases, this study focused on the comparative questions in these canonical texts, while also including comparative questions from the EGMA.

As in the previous study, the isiXhosa texts provide a valuable source of examples of how to formulate comparative questions in isiXhosa, but are not a sufficient source of examples. Because of the small number of examples and because, as mentioned previously, the author is not fluent in isiXhosa, mother tongue isiXhosa speakers were consulted to clarify and exemplify the range of possible formulations of comparative questions in isiXhosa. Before setting up a typology of isiXhosa compare type problems, aspects of isiXhosa grammar that are relevant for the study are discussed.

Relevant isiXhosa grammar

IsiXhosa is a Nguni language spoken by more than 8 million South Africans (of a total of 57 million). The other three Nguni languages spoken in South Africa are isiZulu, isiNdebele and siSwati. As a Bantu language, isiXhosa has many linguistic features that differ substantially from the linguistic features of Indo-European languages. Two features that are relevant for this study are flexibility of word order and a system of concordial agreement.

IsiXhosa word order is not as rigid as English word order. In isiXhosa the most important word in a sentence is emphasised

3.The Curriculum and Assessment Policy Statement (Department of Basic Education 2011) and three sets of learner workbooks produced by the national Department of Basic Education (2018), the National Education Collaboration Trust and the Nelson Mandela Institute.

Comparative question		Problem situation		
			Matching	
	Classic	**No matching**	**1-to-1 embedded**	**1-to-1 not embedded**
	How many more?	*Standard*	-	-
	Alternative	**No matching**	**1-to-1 embedded**	**1-to-1 imposed**
	How many missing?	-	*Roberts (2016)*	-
	How many still needed?	*n/a*	-	*EGMA*
	How many won't get?	-	-	*Hudson (1980)*

EGMA, early grade mathematics assessment; n/a, not applicable.

FIGURE 4: Typology of English compare problems (including examples from literature).

by putting it at the beginning of the sentence. Table 4 shows how the English sentence 'Sigqibo gave Mveli bread' can be constructed in four different ways in isiXhosa, each emphasising different words. This flexibility of word order accounts for some of the variation of isiXhosa compare type problems.

TABLE 4: Example of word order variation possible in isiXhosa.

isiXhosa	English
USigqibo unike uMveli isonka	*Sigqibo* gave Mveli bread
Umnike isonka uMveli uSigqibo	Sigqibo *gave* Mveli *bread*
Isonka uMveli usinike uSigqibo	Sigqibo gave Mveli *bread*
Usinike uMveli isonka uSigqibo	Sigqibo *gave Mveli* bread

Source: Oosthuysen, J.C. (2016). *The grammar of isiXhosa*. Stellenbosch: Sun Media

IsiXhosa, like other Bantu languages, has a noun class system. This means that all nouns belong to a particular class which is determined by the noun's prefix. In a sentence, any word (verb, noun, pronoun or adjective) associated with a noun has to show 'agreement' with that noun. This is achieved by adding a concord (a prefix) to the word, which contains similar-sounding letters to the prefix of the noun. This is referred to as concordial agreement. For example:

Izinja zininzi kuneekati. '	There are more dogs than cats.'
Abantwana baninzi kunoomama.	'There are more children than mothers.'

In the first example, the noun is *izinja* 'dogs' with the prefix *izi-* while in the second sentence the noun is *abantwana* 'children' with the prefix *aba-*. In each sentence the adjective *-ninzi* 'many' takes a different prefix, as determined by the noun it is describing. For this reason, when referring to words on their own (i.e. when they are not referred to as part of a sentence), the root of the word is used (e.g. *-ninzi*) rather than one particular form of the word (e.g. *zininzi* or *baninzi*).

Also relevant for this study is the use of loanwords in isiXhosa. Loanwords are words that are embraced by the speakers of one language (in this case isiXhosa) from another language (the source language) (O'Grady, Dobrovolsky, & Aronoff, 1997). In most cases nouns are borrowed; however, there are some languages that occasionally borrow verbs and adjectives (Brown, 2003). IsiXhosa has many loanwords from English and Afrikaans, most of which are nouns (e.g. *ikati* 'cat'). However, isiXhosa also has a few verb stems that are loanwords from English or Afrikaans. These are used where no isiXhosa words are available and are used in a phonetically adapted form e.g. *-sarha* 'saw' (from Afrikaans 'saag') and *-bhaptiza* 'baptise' (from English) (Oosthuysen, 2016, p. 282).

In the following sections different formulations of isiXhosa comparative questions will be discussed, first in terms of 'classic' comparative questions and then in terms of 'alternative' comparative questions. Finally, a typology of isiXhosa compare type problems will be set up.

'Classic' comparative questions

Unlike English which only has one way to phrase the 'classic' comparative question, in isiXhosa there are a number of different ways in which the 'How many more' question can be phrased (see Figure 5). In this article three commonly used variations are discussed. One reason why variations are possible in isiXhosa is because there are two question words that can be used in combination with two words expressing 'more'. The two question words are *-ngaphi* 'how many' and *kangakanani* 'to what extent'. The two words used to express 'more' are the adjective *-ninzi* 'many/numerous/lots' and the adverb *ngaphezu(lu)* 'more/above' (see Mostert & Roberts, 2020, for a detailed discussion on the use of *-ninzi* and *ngaphezu(lu)* in comparison phrases).

The first formulation of a 'How many more' question in isiXhosa uses *-ngaphi* 'how many' and *ngaphezu(lu)* 'more/above'. In terms of word order, this formulation is the closest to the word order in English. Like in English, the question starts with 'how many' (*-ngaphi*). It is therefore possible that this formulation can result in a similar confusion as in English in that learners might answer the question *Zingaphi iimbiza?* 'How many pots?' instead of *Zingaphi iimbiza ngaphezu kweziciko?* 'How many more pots than lids?'.

The second formulation also uses *-ngaphi* 'how many' but uses *-ninzi* 'many' to express 'more'. In this formulation, and in the third formulation, the adjective *-ninzi* 'many' is used before the question word *-ngaphi* 'how many'. Because the formulation does not start with *-ngaphi* 'how many', it is

Variation	Example
-ngaphi? + ngaphezu(lu)	Zingaphi iimbiza ngaphezulu kweziciko? *zi-ngaphi iimbiza ngaphezulu kwe-ziciko* they.are-how.many pots more compared.to-lids
-ninzi + nga- + -ngaphi?	Iimbiza zininzi ngezingaphi kuneziciko?† *iimbiza zi-ninzi nge-zi-ngaphi kune-ziciko* pots they.are-many by-they.are-how.many compared.to-lids
-ninzi + kangakanani?	Zininzi kangakanani iimbiza kuneziciko? *zi-ninzi kangakanani iimbiza kune-ziciko* they.are-many to.what.extent pots compared.to-lids

†, Alternatively: *Zininzi ngezingaphi iimbiza kuneziciko*? or *Zininzi ngembiza ezingaphi kuneziciko*?

FIGURE 5: Different formulations of the classic 'how many more' comparative question in isiXhosa.

possible that this formulation is less likely to result in learners answering the question 'how many?' instead of the question 'how many more?'.

Note that due to the flexibility of word order in isiXhosa the second formulation can be expressed in a number of different ways. For this study, however, only one variation was considered (see footnote in Figure 5).

The third formulation uses -*ninzi* 'many' and a specialised question word *kangakanani* 'to what extent'. *Kangakanani* can be used when asking about the difference between two nouns or sets of nouns. Because the use of *kangakanani* precludes the use of -*ngaphi*, it was speculated that this third formulation would be the least confusing for learners and therefore the easiest for them to solve. This was tested in this study by comparing the third formulation with one variation of the second formulation.

It is important to note that the question word *kangakanani* 'to what extent' does not necessarily have to have a numerical answer. As in English, when asked, 'How many more stars than squares?', it is possible to answer 'Many more' or 'A few more'. It is therefore important that isiXhosa learners, or at least teachers, are aware that the practice of answering a *kangakanani* question with a numerical value is classroom based and is not necessarily used outside of the mathematics classroom.[4]

The canonical texts contain examples of all three formulations of classic comparative questions. While it is not possible to know what informed the choice of formulation in each example, this study sets out, in part, to provide research to better inform such decisions in the future.

Alternative comparative questions

As discussed previously, in English there are a number of alternative ways of asking comparative questions in conjunction with a matching problem situation. These include 'how many won't get' (Hudson, 1980), 'how many still needed' (EGMA) and 'how many missing' (Roberts, 2016). Similarly it is possible to construct alternative comparative questions in isiXhosa, as exemplified in Figure 6.

4.Thank you to Bambelihle Nkwentsha for pointing this out.

Variation	Example
-shota	Ushota ngeziciko ezingaphi? *u-shota nge-zicko ezi-ngaphi* she.is-short by-lids that.are-how.many She is short by how many lids?
kusafuneka	Kusafuneka iziciko ezingaphi? *ku-sa-funeka iziciko ezi-ngaphi* it.is-still-necessary lids that.are-how.many How many lids are still needed?
ngazukufumana	Zingaphi iimbiza ezingazukufumana iziciko? *zi-ngaphi iimbiza ezi-nga-zu-ku-fumana iziciko* they.are-how.many pots that.are-not-going-to-get lids How many pots won't get a lid?

Note: Words defining the formulation are underlined.

FIGURE 6: Alternative comparative questions in isiXhosa (for matching (one-to-one embedded) problem situations).

While the *kusafuneka* 'still need' and the *ngazukufumana* 'won't get' formulations have direct equivalents in English, the -*shota* formulation does not and therefore requires some additional comments. The adapted loanword -*shota* is a loanword from the English verb 'be short of'. While in English 'be short of' is most commonly used to refer to money (e.g. 'I am short (of) three rand' to mean 'I have three rand less than I need'), in isiXhosa -*shota* is commonly used to refer to being short of a wide range of things. In isiXhosa -*shota* is either used with the prefix *u*- when a person is short of something or with the prefix *ku*- when there is a shortage of things (not belonging to a specific person).

As part of a larger study, isiXhosa adults (both teachers and other caring adults) were observed engaging with isiXhosa learners and formulating comparative questions about specific problem situations. From these observations it appeared that the formulations that learners most easily understood were ones that included the verb -*shota*. This observation was the impetus for this study which, among other things, tests the hypothesis that -*shota* comparative questions are the easiest for learners to solve. Like the 'how many missing?' question introduced by Roberts (2016), asking 'how many are short?' draws attention to the absence of elements.

It is important to note that some isiXhosa speakers argue that it is not appropriate to use a loanword such as -*shota* in a mathematics classroom. Others argue that because it is a word learners are familiar with and understand, it should be used as a means to help learners make sense of compare type problems, at least in teaching, if not in formal testing.

Typology of isiXhosa compare type problems

Drawing on the typology of English comparative questions, and on the discussions of classic and alternative comparative questions in isiXhosa, Figure 7 provides a typology of isiXhosa compare (difference unknown) word problems. As with the English typology, a complete version of the typology with examples for each category is provided in Appendix 1, Figure 2-A1. The typology and the identification of the dimensions of variation (problem situation and comparative question) and the range of possible change for each dimension

Comparative question	Classic		Problem situation: No matching	Matching: 1-to-1 embedded	Matching: 1-to-1 not embedded
	ngaphezu(lu) + -ngaphi?	more + how many?	-	-	-
	-ninzi + nga + -ngaphi?	many + by + how many?	Q1	Q3	-
	-ninzi + kangakanani?	many + to what extent?	Q4	-	-
	Alternative		**No matching**	**1-to-1 embedded**	**1-to-1 imposed**
	ushota + -ngaphi?	'short' + how many?	-	Q2	-
	kusafuneka + -ngaphi?	still necessary + how many?	*n/a*	-	Q5
	ngazukufumana + -ngaphi?	won't get + how many?	-	-	-

n/a, not applicable.

FIGURE 7: Typology of isiXhosa compare type problems (with early grade mathematics assessment questions located in relevant categories).

make it possible to study the influence of one dimension by varying that dimension and keeping the other dimension the same.

Research design

In order to answer the two research questions, as set out in the introduction, results from the South African version of the EGMA, based on the core EGMA (Platas, Ketterlin-Geller, Brombacher, & Sitabkhan, 2014) and adapted by Brombacher and Associates, were used. The core EGMA includes one compare type problem out of a total of four additive relation word problems (Q5–Q8 in Table 5). For this study, four additional compare type word problems were also administered (Q1–Q4 in Table 5).

The four original word problems were translated from English into isiXhosa by an accredited translator. The additional four problems were formulated in isiXhosa through consultation with a number of isiXhosa speakers. See Appendix 1, Table 1-A1 for the isiXhosa formulation and English translations of the eight word problems.

Table 5 sets out the eight different types of word problems used in this study, based on the typologies set up in previous sections.

The research for this study was done in two stages, corresponding to the two research questions. In the first stage, two additional compare type problems (Q1 and Q2) were added to the EGMA assessment in order to answer the first research question, namely whether, in isiXhosa, different formulations of compare type problems had different levels of difficulty. This was answered by comparing Q1, Q2 and Q5. The results of this comparison (discussed below) confirmed that in isiXhosa, different formulations of compare type problems have different levels of difficulty. However, at this point it became apparent that the three formulations tested in the first stage differed in terms of more than one dimension. This led to stage two of the study in which the second research question was answered.

TABLE 5: Different word problems used in this study.

Variable	Question	Problem type	Stage 1 (n = 242)	Stage 2 (n = 260)
Compare	Q1	No matching [*-ninzi + -ngaphi*]	x	-
	Q2	Matching (1-to-1 embedded) [*-shota*]	x	-
	Q3	Matching (1-to-1 embedded) [*-ninzi + -ngaphi*]	-	x
	Q4	No matching [*-ninzi + kangakanani*]	-	x
	Q5	Matching (1-to-1 imposed) [*kusafuneka*]	x	x
Other	Q6	Change (increase) – result unknown	x	x
	Q7	Change (increase) – start unknown	x	x
	Q8	Collection (attributes) – subset unknown	x	x

In stage two, in order to establish which of the different dimensions had an effect on the difficulty level, Q1 and Q2 were replaced with Q3 and Q4. The addition of Q3 and Q4 made it possible to isolate the effect of the comparative question (research question 2.1) by comparing two differently phrased questions with the same problem situation (Q1 and Q4 as well as Q2 and Q3). It also meant that it was possible to isolate the effect of the problem situation (research question 2.2) by comparing two problems with the comparative question formulated in the same way but with different problem situations (Q1 and Q3). The relationship between the four questions and where they are located in the typology of isiXhosa compare type problems is set out in Figure 8.

Methodology

In this section the methodologies used for data collection and data analysis are discussed in detail.

Data collection

The EGMA was administered to isiXhosa-speaking children in Grade 1, Grade 2 and Grade 3. The data were originally collected to evaluate an early grade mathematics intervention in five isiXhosa-dominant public schools in the rural Eastern

			Problem situation	
			No matching	Matching (1-to-1 embedded)
Comparative question	*-ninzi + nga + -ngaphi?*	many + by + how many?	Q1	Q3
	-ninzi + kangakanani?	many + to what extent?	Q4	-
	-shota + -ngaphi?	'short' + how many?	-	Q2

FIGURE 8: Questions used to isolate effect of different dimensions of variation.

Cape. All learners who were present on the day that the assessment was administered were tested.

The EGMA was administered twice in 2019, once in May (Stage 1, n = 242) and once in November (Stage 2, n = 260). Q1 and Q2 were added to the EGMA administered in May and Q3 and Q4 were added to the EGMA administered in November. Table 5 also indicates which questions were administered during each stage.

The EGMA was administered individually by isiXhosa-speaking adults. Each word problem was read to the learner in isiXhosa, first using isiXhosa number names, and then using English number names. Results (correct or incorrect) were recorded on tablets and then extracted into a spreadsheet for analysis.

The guidelines for administering the EGMA state that if a learner incorrectly answers four questions in a row, they should not be asked the remaining questions in that section, the assumption being that the learner would not be able to answer any of the remaining questions. For this article, in each stage, only the results of learners who were asked all six word problems were analysed.

Data analysis

In order to compare the difficulty level of two word problems, the facility score of each problem was calculated. The facility score is the percentage of learners who correctly answered the question out of the total number of learners who were asked the question. Questions with a higher facility score were considered to be easier than those with a lower facility score.

Because neither cohort answered all five compare type questions (see Table 5), it was necessary to establish whether the results of the questions administered only in Stage 1 could be compared with the questions administered only in Stage 2, and with those administered during both stages. In order to do this a Pearson's chi-squared test for homogeneity was done on facility scores of the four questions that were administered during both stages (Q5–Q8 in Table 5). The test returned a p-value of 0.96 indicating that the two groups of learners were very homogenous, in other words the learners performed similarly on the four matched questions. In light of this, it is possible to compare any two questions, even if they were not answered by the same group of learners.

While early research on the relative difficulty of word problems only considered the facility scores of the problems, subsequent developments in data analysis techniques now allow for more sophisticated comparisons. In this study, for each research question the word problems were first compared in terms of their facility scores. If there was a difference in facility score, a Pearson's chi-squared test for independence was used to establish whether the difference in facility score was significant or not (see Appendix 1, Table 2-A1 for a summary of the p-values for each research question).

Ethical consideration

This study forms part of a PhD study which has received ethical approval from the University of Johannesburg, ethical clearance number: 2017-060. The data were anonymised and used only as aggregated data, which were not linked to individual children. Ethical clearance was received on 08 September 2017.

Results

The results will be discussed in relation to each research question. For each question the relevant problems will first be compared in terms of their facility scores and then in terms of the results of the Pearson's chi-squared test for independence.

RQ1: Relative difficulty of different isiXhosa compare type problems

In this section three different types of comparison problems are compared, namely a standard compare type problem (Q1), a matching (one-to-one embedded) problem (Q2), and a matching (one-to-one imposed) problem (Q5):

(Q1) Iiswiti zentombazana zininzi ngeeswiti ezingaphi kwezenkwenkwe?

iiswiti	*ze-ntombazana*	*zi-ninzi*	*ngee-switi*	*ezi-ngaphi*
sweets	of-girl	they.are-many	in.terms. of-sweets	they.are-how. many

kwe-ze-nkwenkwe

compared.to-of-boy

How many more sweets does the girl have than the boy?

(Q2) Ushota ngeziciko ezingaphi?

u-shota	*nge-ziciko*	*ezi-ngaphi*
she.is-short	in.terms.of-lids	they.are-how.many

She is short by how many lids?

(Q5) Kusafuneka iiorenji ezingaphi ukuze umama akwazi ukunika umntwana ngamnye iorenji enye?

ku-sa-funeka	*iiorenji*	*ezi-ngaphi*	*ukuze*	*umama*	*a-kwazi*
it.is-still-necessary	oranges	that.are-how.many	so.that	mother	she-is.able
uku-nika	*umtwana*	*nga-mnye*	*iorenji*	*e-nye*	
to-give	child	by-one	orange	that.is-one	

How many oranges are still needed so that the mother can give each child one orange?

Figure 9 shows that, like in English, a smaller proportion of learners (47%) were able to solve standard compare type problems ('no matching' problem situation with a classic phrasing of the compare question) than both of the 'matching' problems (Q2 and Q5). There was less of a difference in the relative difficulty of the two different matching problems with a bigger proportion of learners able to answer the matching (one-to-one imposed) problem correctly (81%) than learners who answered the matching (one-to-one embedded) problem correctly (74%). The difference in facility scores was significant ($p < 0.05$).

RQ2: Effect of different factors on relative difficulty of word problems

In this section the different factors or dimensions that constitute a compare type problem are isolated to establish which factors influence the relative difficulty of compare type problems. The two factors that are considered are the problem situation and the phrasing of the comparative question. This is done by comparing different combinations of Q1–Q4, as outlined in the research design section.

Figure 10 shows that, for matching (one-to-one embedded) problems, a *-shota* formulation of the comparative question is easier for learners than a *-ninzi* + *-ngaphi* formulation. It also shows that, for no matching problems, a *kangakanani* formulation is easier than a *-ninzi* + *-ngaphi* formulation. Finally, when a *-ninzi* + *-ngaphi* formulation is used, problems with a matching (one-to-one embedded) problem situation are easier than those with a 'no matching' problem situation. These results are discussed in detail in the following two sections.

RQ2.1: Effect of formulation of comparative question on relative difficulty

To investigate whether the formulation of the comparative question influences the relative difficulty of a word problem, two different problem types were considered: matching (one-to-one embedded) problems and standard compare type problems. For each problem type the same problem situation was used but two differently phrased questions were asked.

For matching (one-to-one embedded) problems, one question used the classic '*-ninzi* + *-ngaphi*' phrasing (Q3) and the other the alternative '*-shota*' phrasing (Q2):

(Q3) Iimbiza zininzi ngezingaphi kuneziciko?

iimbiza zi-ninzi nge-zi-ngaphi kune-ziciko

pots they.are-many by-they.are-how.many compared.with-pots

How many more pots than lids?

(Q2) Ushota ngeziciko ezingaphi?

u-shota nge-ziciko e-zi-ngaphi

she.is-short in.terms.of-lids are-they-how.many

How many lids are short (missing)?

Figure 10 shows that fewer learners answered correctly when the '*-ninzi* + *-ngaphi*' phrasing was used (59%) and more learners answered correctly when the '*-shota*' phrasing was used (74%). The difference in facility scores is significant ($p < 0.05$). This comparison is similar to the comparison Hudson (1980) tested. Like with Hudson's comparison, it is possible that the alternative '*-shota*' phrasing is easier for learners to understand because it does not use quantifiers such as 'more' and 'less'.

For the 'no matching' compare problems, it was possible to compare two different formulations of the classic comparative

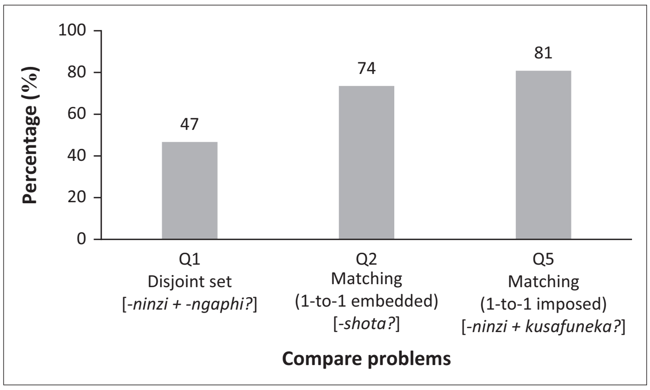

FIGURE 9: Facility score of three different isiXhosa compare type problems.

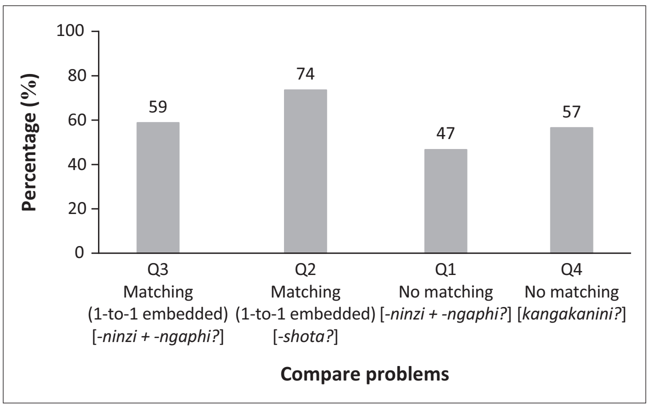

FIGURE 10: Facility scores of problems used to isolate factors influencing relative difficulty.

questions, namely the '*-ninzi + -zingaphi*' formulation (Q1) and the '*kangakanani*' formulation (Q4). As can be seen from the English translations, this comparison is not possible in English where there is only one formulation of the 'classic' comparative question.

(Q1) Iiswiti zentombazana zininzi ngeeswiti ezingaphi kwezenkwenkwe?

iiswiti ze-ntombazana zi-ninzi ngee-switi ezi-ngaphi

sweets of-girl they.are-many in.terms.of-sweets they.are-how.many

kwe-ze-nkwenkwe

compared.to-of-boy

How many more sweets does the girl have than the boy?

(Q4) Zininzi kangakanani iiswiti zentombazana kunezenkwenkwe?

zi-ninzi kangakanani iiswiti ze-ntombazana kune-ze-nkwenkwe?

they.are-many by.what.extent sweets of-girl compared.to-of-boy?

How many more sweets does the girl have than the boy?

Figure 10 shows that fewer learners answered correctly when the '*-ninzi + -ngaphi*' formulation was used (47%) and more answered correctly when the '*kangakanani*' formulation was used (57%). This supports the hypothesis that learners would find the '*kangakanani*' formulation less confusing than the '*-ninzi + -ngaphi*' formulation and begins to answer the question about which formulation of the 'classic' comparative question is most accessible to learners. Even though facility scores for these two problems are not that different, the difference is still significant ($p = 0.003 < 0.05$).

RQ2.2: Effect of problem situation on relative difficulty of word problem

In order to establish whether reframing the problem situation without changing the question also has an effect on the relative difficulty level of compare problems, the facility scores for a standard compare problem (Q1) and a matching (one-to-one embedded) problem (Q3), both with the same formulation of the 'classic' comparison question (*-ninzi + -ngaphi*), were compared:

(Q1) Iiswiti zentombazana zininzi ngeeswiti ezingaphi kwezenkwenkwe?

iiswiti ze-ntombazana zi-ninzi ngee-switi ezi-ngaphi

sweets of-girl they.are-many in.terms.of-sweets they.are-how.many

kwe-ze-nkwenkwe

compared.to-of-boy

How many more sweets does the girl have than the boy?

(Q3) Iimbiza zininzi ngezingaphi kuneziciko?

iimbiza zi-ninzi nge-zi-ngaphi kune-ziciko

pots they.are-many by-they.are-how.many compared.with-pots

How many more pots are there than lids?

Figure 10 shows that even when a more difficult 'classic' comparative question is used, a higher percentage of learners correctly answered the matching (one-to-one embedded) question (59%) than the percentage of learners who correctly answered the no matching problem (47%). This suggests that changing the problem situation and not the question can, on its own, make it easier for learners to understand the problem. Again, even though there is not a big difference between the facility score of these two questions, the chi-squared test ($p = 0.006 < 0.05$) confirms that the difference in facility score is significant.

Discussion

These results raise a number of points regarding the relative difficulty of isiXhosa compare type problems in early grade mathematics, some of which are also relevant for English. The study confirms that in isiXhosa, as in English, while standard compare type problems (no matching + classic comparative question) are the most difficult to solve, when a standard compare type problem is modified, either by changing the problem situation or the formulation of the comparative question, the problem can become significantly easier for learners to solve (see Figure 11).

The next two points are only relevant for isiXhosa as they relate to specialised words that do not have an English equivalent.

Classroom observations suggested that comparative questions using the loanword *-shota* (e.g. Q2) would be easier for learners to understand those using *-ninzi* and *-ngaphi* (e.g. Q3). This was confirmed by the results from this study.

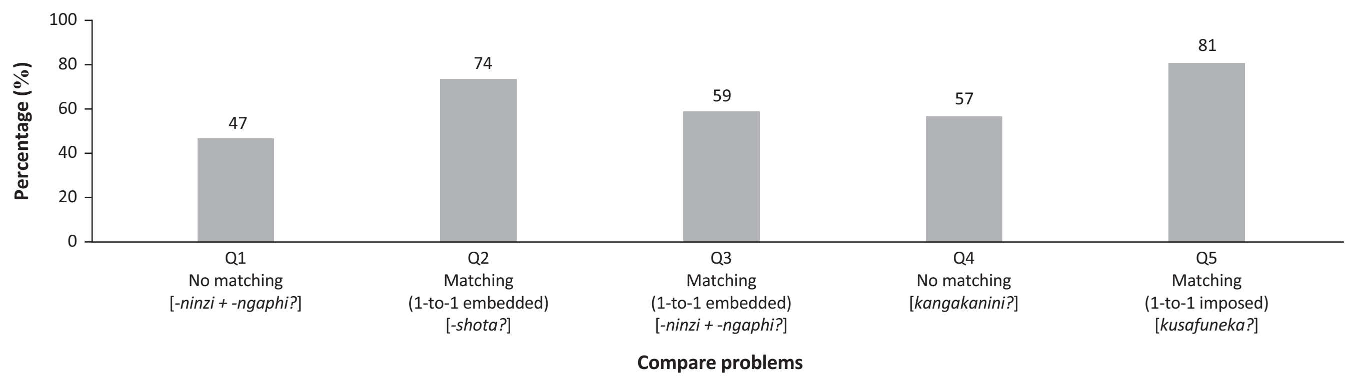

FIGURE 11: Facility score of different compare type word problems.

It is possible that this difference in difficulty level is because the '*-shota*' formulation does not use the quantifier 'more' while the '*-ninzi* + *-ngaphi*' formulation does. This difference in facility score suggests that teachers can use the '*-shota*' formulation to introduce learners to compare type problems.

It was speculated that the specialised question word *kangakanani* 'to what extent' would be less confusing for learners than questions using *-ninzi* and *-ngaphi* as these could be confused with *-ngaphi* questions. The results confirm this speculation: a higher percentage of learners correctly answered the *-kangakanani* question (Q4) than those that correctly answered the '*-ninzi* + *-ngaphi*' question (Q1) when the problem situation was kept the same. The fact that isiXhosa has a specialised question word that can be used when asking about the difference between two nouns or sets of nouns is an affordance that could be leveraged to mitigate the possible confusion between 'How many?' questions and 'How many more?' questions.

The final two points relate to issues that are also applicable in English.

While both matching (one-to-one embedded) and matching (one-to-one imposed) problem situations provide a useful teaching tool, in terms of the relative difficulty of comparison problems, matching (one-to-one imposed) problems are slightly easier to solve (see Figure 11). One possible explanation for this is that in matching (one-to-one imposed) problems (e.g. Q5) an additional phrase such as 'each child can get one orange' is required. This additional phrase makes the matching action explicit while in the matching (one-to-one embedded) problem (e.g. Q2), the matching action is implicit.

Finally, the influence of matching problem situations is not only observed when used together with alternative formulations of the comparative question. When a more difficult classic comparative question is used with both a 'no matching' problem situation (e.g. Q1) and with 'matching' problem situation (e.g. Q3), the problem with the 'matching' situation is still easier for learners to solve than the problem with the 'no matching' situation. It is possible that the reason for this is because the matching problem situation invokes the action of matching which can be used to solve the problem.

There are a number of limitations to be taken into account when interpreting the results. In order to fully establish whether there is a difference between matching (one-to-one embedded) and matching (one-to-one imposed) problems, it would be necessary to compare the two problem situations using all six different types of question. Similarly, in order to establish whether there is a difference between a '*-zingaphi*' question and a '*kangakanani*' question, it would be necessary to compare all three different compare type problems (no matching, matching (one-to-one embedded), matching (one-to-one imposed)). Finally, because the study did not compare all three classic comparative problems that can be formulated in isiXhosa, it is not possible, at this stage, to establish which formulation is easiest for learners to understand.

Conclusion

Compare type word problems are notoriously difficult for learners but are also an important opportunity for learners to engage with the notion of comparison and of 'subtraction as difference'. While the 'standard' formulation of compare problems (no matching problem situation with classic comparative question) is difficult, this and other studies have shown that certain formulations of compare type problems are easier for learners to understand and to solve, both in English and in isiXhosa. These easier formulations provide a means of accessing the 'standard' compare type problems, allowing learners to make meaning of the problem situation without having to navigate complex language.

This article has contributed to understanding the different factors that constitute a compare type (difference unknown) word problem. The typologies of English and isiXhosa compare type problems provide a resource that can be used by materials developers and in further research looking in more detail at the influence of the different factors.

This article also highlights the importance of studying the ways in which African languages express mathematical ideas in order to identify and leverage affordances for teaching and learning mathematics and, where different formulations are possible, to establish which formulation is most accessible for learners. While this article lays a foundation for studying compare type problems in other Nguni languages, ultimately such research needs to be led by home language speakers in order for the linguistics features of African languages to be explored and described on their own terms, and not primarily in relation to English.

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Competing interests

The author has declared that no competing interest exists.

Author's contributions

I declare that I am the sole author for this article.

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Data availability statement

The data that support the findings of this study are available on request from the corresponding author.

Disclaimer

The views and opinions expressed in this article are those of the author and do not necessarily reflect the official policy or position of any affiliated agency of the author.

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Appendix starts on the next page →

Appendix 1

		Problem situation		
		No matching	Matching	
			1-to-1 embedded	1-to-1 not embedded
Comparative questions	**Classic**	A girl has seven sweets. A boy has five sweets.	A mother has eight pots and five lids.	A mother has seven children and two oranges.
	how many more?	How many more sweets does the girl have than the boy?	How many more pots than lids?	How many more children than oranges?
	Alternative	**No matching**	**1-to-1 embedded**	**1-to-1 imposed**
	how many missing?	-	How many lids are missing?	How many oranges are missing if the mother wants to give each child one orange?
	how many still needed?	*n/a*	How many lids are still needed?	How many oranges are still needed if the mother wants to give each child one orange?
	how many won't get?	-	How many pots won't get a lid?	How many children won't get an orange if the mother gives each child one orange?

Note: Grey highlighted text indicates phrase needed to impose one-to-one matching. n/a, not applicable.

FIGURE 1-A1: Typology of English compare word problems.

		Problem situation		
		No matching	Matching	
			1-to-1 embedded	1-to-1 not embedded
Comparative questions	**Classic**	Intombazana ineeswiti ezilithoba. Inkwenkwe ineeswiti ezine.	Umama uneembiza ezisibhozo neziciko ezihlanu.	Umama unabantwana abasixhenxe, abe neeorenji ezimbini.
	ngaphezu(lu) + -ngaphi?	Zingaphi iiswiti zentombazana ngaphezu kwe zenkwenkwe?	Zingaphi iimbiza ngaphezu kweziciko?	Bangaphi abantwana ngaphezu kweeorenji?
	-ninzi + nga + -ngaphi?	Iiswiti zentombazana zininzi ngeeswiti ezingaphi kwezenkwenkwe?†	Iimbiza zininzi ngezingaphi kuneziciko?†	Abantwana baninzi ngezingaphi kuneeorenji?
	-ninzi + kangakanani?	Zininzi kangakanani iiswiti zentombazana kunezenkwenkwe?†	Zininzi kangakanani iimbiza kuneziciko?	Baninzi kangakanani abantwana kuneeorenji?
	Alternative	**No matching**	**1-to-1 embedded**	**1-to-1 imposed**
	ushota + -ngaphi?	-	Ushota ngeziciko ezingaphi?†	Ushota ngeorenji ezingaphi ukuze umama akwazi ukunika umntwana ngamnye iorenji enye?
	kusafuneka + -ngaphi?	*n/a*	Kusafuneka iziciko ezingaphi?	Kusafuneka iiorenji ezingaphi ukuze umama akwazi ukunika umntwana ngamnye iorenji enye?†
	ngazukufumana + -ngaphi?	-	Zingaphi iimbiza ezingazukufumana iziciko?	Bangaphi abantwana abangazukufumana ukuba umama unika umtwana ngamnye iorenji enye?

Note: †, indicates problems used in study. Grey indicates phrase needed to impose one-to-one matching. n/a, not applicable.

FIGURE 2-A1: Typology of isiXhosa compare word problems.

TABLE 1-A1: Compare problems used in study.

Question	Description
Q1	*Inkwenkwe ineeswiti ezine. Intombazana ineeswiti ezilithoba. Iiswiti zentombazana zininzi ngeeswiti ezingaphi kwezenkwenkwe?* A boy has four sweets. A girl has nine sweets. How many more sweets does the girl have than the boy?
Q2	*Umama uneembiza ezisibhozo neziciko ezihlanu. Ushota ngeziciko ezingaphi?* A mother has eight pots and five lids. How many lids is she short?
Q3	*Umama uneembiza ezisibhozo neziciko ezihlanu. Iimbiza zininzi ngezingaphi kuneziciko?* A mother has eight pots and five lids. How many more pots are there than lids?
Q4	*Inkwenkwe ineeswiti ezine. Intombazana ineeswiti ezilithoba. Zininzi kangakanani iiswiti zentombazana kunezenkwenkwe?* A boy has four sweets. A girl has nine sweets. To what extent are the girl's sweets more than the boy's sweets?
Q5	*Umama unabantwana abasixhenxe, abe neeorenji ezimbini. Kusafuneka iiorenji ezingaphi ukuze umama akwazi ukunika umntwana ngamnye iorenji enye?* A mother has seven children, and she has two oranges. How many oranges are still needed so that the mother can give the children each one orange?

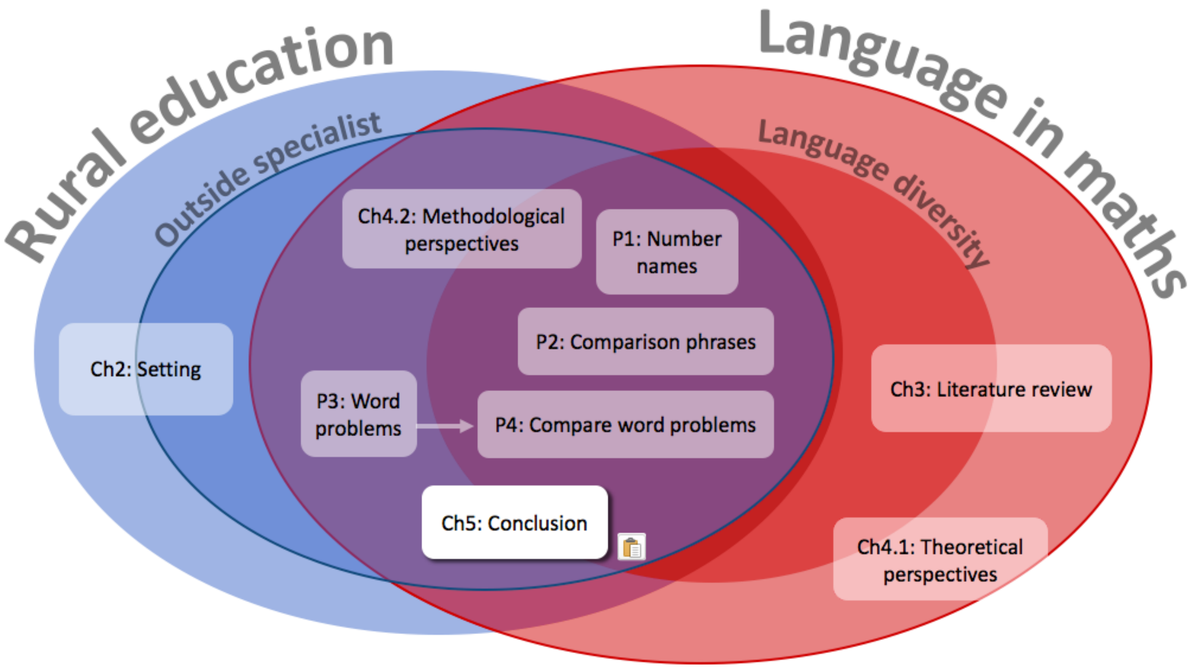
TABLE 2-A1: P-values of problems that were compared for each research question.

RQ	Questions compared	Dimension compared	p
1	Q1, Q2, Q5	n/a	< 0.001*
2.1	Q1, Q4	different questions (same 'no matching' situation)	0.030*
2.1	Q2, Q3	different questions (same 'matching' situation)	< 0.001*
2.2	Q1, Q3	different problem situations (same '-ngaphi' question)	0.006*

n/a, not applicable.

*, $p < 0.05$ indicating significant differences in facility scores.

CHAPTER 5: CONCLUSION



5.1 Overview

This study set out to contribute to addressing the problem behind the problem of South Africa's mainstream education system which is still failing the majority of learners. According to Ramadiro and Porteus (2017), the problem behind the problem is that the knowledge and expertise of education specialists is not embedded in, nor held accountable to, the realities of the majority of South African schools – particularly rural schools. One of these realities is the linguistic profile of rural schools. In most rural schools, all learners have the same indigenous African home language, a language they share with their teachers and, in the Foundation Phase, this African language is also the language of instruction. However, because of a lack of expertise within the system, the potential of home language instruction has not been realised (Ramadiro & Porteus, 2017). This is evident from learner results and teachers' lack of proficiency in the mathematics register of the language of instruction, in this case, isiXhosa.

Through a series of four papers, this study contributes to building knowledge in the field of mathematics education which is accountable to the realities of rural schools.

As a study made up of four papers that are knitted together through a common theme, this study has a *central* research question which is answered by each paper individually, and it has an *overarching* research question answered by the four papers as a whole. As a researcher who is not embedded within rural isiXhosa-dominant schools, and who is not proficient in isiXhosa, I am limited in terms of what I can research. This is captured in the overarching question:

> *What research possibilities are available for an "outside specialist" wishing to build the education knowledge project in ways that are accountable to rural isiXhosa-dominant schools?*

While each paper has its own specific research question, each paper also contributes to answering the question at the centre of the study, namely:

> *What are the affordances and constraints of teaching early grade mathematics in rural isiXhosa-dominant classrooms, both in relation to the linguistic features of isiXhosa and the instructional practices of teachers?*

Because of my linguistic constraints as an outside specialist who is not proficient in isiXhosa, and because of the affordances provided by the large corpus of translated canonical texts available in the South African education system, each paper takes as its unit of analysis a collection of examples from a set of canonical texts.

I begin this chapter by discussing each of the four papers individually. I first consider how each paper employed the theoretical and methodological perspectives set out in Chapter 3. Then I show how each paper contributed to answering the central research question.

After discussing the papers individually I discuss the contributions, limitations and implications of the collection of papers as a whole. The contributions are both theoretical and methodological while the limitations relate to my positionality as an outside specialist. The implications include implications for learning and teaching, for the design of instructional materials, and for further research. Finally, I consider the extent to which this collection of papers and the discussion of the papers has answered the overarching research question.

5.2 Employing theoretical and methodological perspectives

In this section I consider how each of the four papers employed the theoretical and methodological perspectives presented in Chapter 4. I discuss how the ideas from the theoretical framework, namely the notion of a dimension of variation (Marton & Booth, 1997) and of the range of change (Watson & Mason, 2006) were applied, and how the methodological approach of creating and populating an example space were used (see Figure 5.1).

As outlined in Chapter 3, my methodological approach included:

1) identifying examples in canonical texts;
2) identifying the dimensions in which they vary;
3) identifying the range of change for each dimension;
4) setting up a typology with different dimensions and range of change;
5) populating the typology with existing examples;
6) identifying missing examples in the typology; and
7) generating missing examples (working with isiXhosa-speakers to generate examples missing from the isiXhosa typologies).

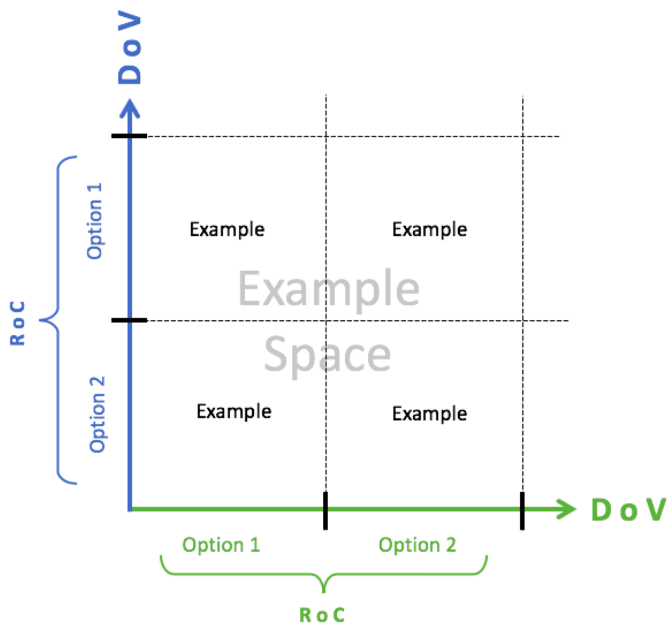


Figure 5.1: Example space generated by changing dimension of variation

Where applicable, I also discuss how the notion of language as a dimension of variation (see Figure 5.2) was employed to better understand the constraints and affordances of how two different languages express mathematical ideas, and how this deepens our understanding of these mathematical ideas (Edmonds-Wathen, 2019). For these cases, I discuss the tools used to compare the two languages, for example, by using direct translation or interlinear morphemic glossing (IMG).

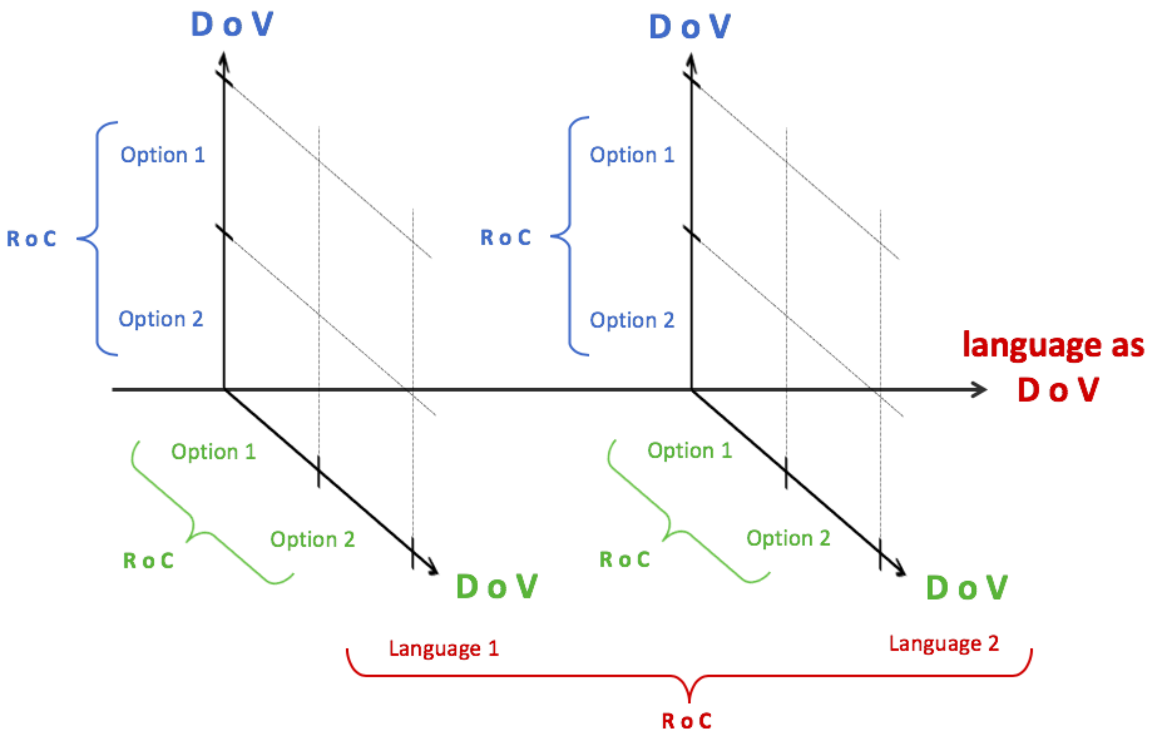


Figure 5.2: Example space with language as dimension of variation

For the cases in which I was comparing languages, the final step of my methodological approach included:

8) for a particular example, keeping the example the same and only varying the dimension of language.

5.2.1 Paper 1: Number names: Do they count?

The first paper focused on isiXhosa and English number names and set out to answer the following questions:

- What are the linguistic features of isiXhosa that affect the articulation of numbers in isiXhosa, and how do they affect them?
- What affordances and constraints do these linguistic features provide for the learning and teaching of numbers in early grade mathematics?

The collection of examples used in this study (in other words, the unit of analysis) was isiXhosa and English number names for whole numbers from one to 100. In line with the cross-cutting methodology, the first step taken was to collect examples of number names from canonical texts such as the DBE, NECT and NMI workbooks as well as from the verbal forward number word sequence used in classrooms for counting. This collection of examples is shown in Table 5.1.

Table 5.1: Examples of isiXhosa number names in different canonical texts

	Numeral nouns	Noun class	Adjectival stem or relative	FNWS	DBE workbooks	NMI workbooks	NECT workbooks
1	isinye	7/8[1]	-nye	inye	inye	nye	nye
2	isibini	7/8	-bini	(zi)mbini	zimbini	mbini	mbini
3	isithathu	7/8	-thathu	(zi)ntathu	zintathu	ntathu	ntathu
4	isine	7/8	-ne	(zi)ne	zine	ne	ne
5	isihlanu	7/8	-hlanu	(zi)ntlanu	zintlanu	ntlanu	ntlanu
6	isithandathu	7/8	-thandathu	(zi)ntandathu	zintandathu	ntandathu	zintandathu
7	isixhenxe	7/8	sixhenxe	(zi)sixhenxe	zisixhenxe	sixhenxe	zisixhenxe
8	isibhozo	7/8	sibhozo	(zi)sibhozo	zisibhozo	sibhozo	zisibhozo
9	ithoba	5/6	lithoba	(zi)lithoba	zilithoba	lithoba	zilithoba
10	ishumi	5/6	lishumi	(zi)lishumi	zilishumi	ishumi	zilishumi

[1] Most classes in isiXhosa occur in a singular/plural pair so that the nouns in class 8 are the plurals of the nouns in class 7 and, unlike English, it is the prefix of a noun that indicates whether it is singular or plural.

Source: Paper 1 (Number names), Table 1

The next step was to identify the dimensions along which the examples varied. In this case the dimensions were a function of the decimal system. If "the way in which a ten is expressed" is taken as one dimension, and "the way in which a one is expressed" as another dimension, Table 5.2 can be seen as an example space of isiXhosa number names.

Table 5.2: Example space of isiXhosa number names

1	*inye*	11	*ishumi elinanye*	21	*amashumi amabini ananye*
2	*(zi)mbini*	12	*ishumi elinesibini*	22	*amashumi amabini anesibini*
3	*(zi)ntathu*	13	*ishumi elinesithathu*	23	*amashumi amabini anesithathu*
4	*(zi)ne*	14	*ishumi elinesine*	24	*amashumi amabini anesine*
5	*(zi)ntlanu*	15	*ishumi elinesihlanu*	25	*amashumi amabini anesihlanu*

Source: Table 2 from Paper 1 (Number names)

While not included in Paper 1 (Number names), Table 5.3 is the equivalent example space for English number names.

Table 5.3: Example space of English number names

1	*one*	11	*Eleven*	21	*twenty one*
2	*two*	12	*Twelve*	22	*twenty two*
3	*three*	13	*Thirteen*	23	*twenty three*
4	*four*	14	*Fourteen*	24	*twenty four*
5	*five*	15	*Fifteen*	25	*twenty five*

Because the Table 5.2 and 5.3 are structured using the two dimensions, by moving along a row or column, one dimension is fixed while the other is varied. This is an example of "separation", one of the significant patterns of variation identified in variation theory (Marton & Pang, 2006). By moving along either a row or a column, it is possible to see the effect of each dimension on the regularity, or irregularity, of the number naming system. Even though Paper 1 (Number n did not include the English Table 5.3, the paper still demonstrated that isiXhosa number names are more regular and more transparent than English number names, while English number names are shorter than isiXhosa number names.

The next step, identifying and generating missing examples, was not applicable in this paper as the examples were number names and there were therefore no missing cases. Because both isiXhosa and English number names were considered, it was possible to consider language as a dimension of variation. This was done by keeping the other dimensions fixed and only varying the language. In other words, by choosing a specific two digit number and comparing the way in which isiXhosa and English expressed this number. Because I had not yet come across interlinear morphemic glossing (IMG) (Comrie et al., 2008) at the time I wrote the first paper, I did not use formal IMG to do this comparison.

Instead I used direct translation as demonstrated in example (1). I laid out the translation in a manner which turned out to be similar to the second and third line of an interlinear morphemic glossing:

(1) **ama**shumi **ama**thathu **ane**sixhenxe
tens that are three that are with seven

While this approach did highlight the similarities and differences between isiXhosa and English number names, it was not as clear as an interlinear morphemic gloss would have been. For example, I was using **bold** to show prefixes and underline to show which parts of the isiXhosa words corresponded to English numbers. An interlinear morphemic glossing would have included four lines and clearly indicated the different morphemes, as demonstrated in example (2):

(2) amashumi amathathu anesixhenxe
amashumi *ama-thathu* *a-ne-sixhenxe*
tens that.are-three that.are-with-seven
'thirty seven'

Even though formal IMG was not used, the approach I used in Paper 1 (Number names) made it possible to identify the affordances and constraints of the way in which each language expresses two-digit number names.

5.2.2 Paper 2: Diversity of mathematical expression: The language of comparison in English and isiXhosa early grade mathematics texts

The second paper considered comparison phrases and set out to answer the following questions:

- What comparison phrases are early grade learners expected to be familiar with, according to the English version of canonical South African mathematics texts?
- How do English and isiXhosa comparison phrases differ, as exemplified in canonical South African mathematics texts?

In line with the methodological approach outlined in Chapter 3, the initial step in answering these two questions was to collect examples of comparison phrases from four early grade canonical mathematics texts, written in English and translated into isiXhosa. In this case the canonical texts were the CAPS and the DBE, NECT and NMI workbooks.

The following step was to identify the dimensions along which the examples varied. Rather than using existing dimensions, like the ones built into the decimal system as in Paper 1 (Number names) or ones already identified in the literature, as in Paper 3 (Distribution of word problems), for this paper I relied on a bottom up approach, being guided by the variation that I observed in the examples in the canonical texts in order to identify dimensions of variation. These dimensions were later confirmed and refined based on insights from linguistics, for example, from the work by Kennedy (2009) on comparison. Note that while Kennedy (2009) identified the ordering of superiority/inferiority and the standard against which the comparison is being made as the two linguistic components of comparisons; I extended this in Paper 2 (Comparison phrases) to include "the difference" as a third linguistic component of comparisons in terms of numerosity.

The three dimensions of variation of comparison phrases identified in the examples in canonical texts were: what is being compared (nouns, objects belonging to people or numbers); whether the referent was implicit or explicit; and whether "the difference" was quantified or not. The dimensions were used to set up an example space, or typology, of English comparison phrases which was then populated with examples from the canonical texts. Missing examples (in other words, categories that did not have examples from the canonical texts) were generated and indicated by italics (see Table 5.4).

Table 5.4: Example space of different types of comparison phrases in English

Category	Difference	**Implicit referent** [more as "additional"]	**Explicit referent** [more _ than _]
Nouns (objects & people)	**Not quantified**	Draw more circles.*	Are there more circles or squares? There are more circles than squares. *
	Quantified	*How many more circles?* Draw 3 more circles.*	How many more circles are there than squares? *There are 3 more circles than squares.**
Objects (belonging to people)	**Not quantified**	*Sbu gets/needs more sweets.*	Does Sbu or Sive have more sweets? *Sbu has more sweets than Sive.*
	Quantified	How many more sweets does Sbu get/need? Sbu gets/needs 3 more sweets.	How many more sweets does Sbu have than Sive? Sbu has 3 more sweets than Sive.
Numbers	**Not quantified**		Which is more, 8 or 5? 8 is more than 5.*
	Quantified		*How much more is 8 than 5?* 8 is 3 more than 5.*
	Quantified		How much is 3 more than 5? 3 more than 5 is 8.

** examples used in the simplified isiXhosa typology*

Source: Table 1 in Paper 2

Because the canonical texts were available in English and isiXhosa, it was possible to set up a typology of isiXhosa comparison phrases and compare these with the English comparison phrases, thereby making it possible to view language as a dimension of variation. However, this was not a simple process.

While English comparison phrases only vary along three dimensions, isiXhosa comparison phrases vary along a fourth dimension, namely the way in which an ordering of superiority is expressed. English only uses the word *more* to express an ordering of superiority (and in the case of numbers sometimes *bigger*, *greater* or *larger*) while isiXhosa uses different adjectives as well as the adverb *-ngaphezulu* 'more/above'[10].

In order to compare English and isiXhosa comparison phrases, a simplified typology was used. The simplified typology did not include the comparison of objects belonging to people nor the empty implicit referent (numbers) category and only included comparison phrases, not comparison questions. This made it possible to collapse the *implicit/explicit referent* dimension

[10] As discussed in Paper 2, isiXhosa comparison phrases sometimes also use specific verbs or words implying an ordering of superiority. These were included in the paper for completeness but are excluded from this discussion as they were not part of the typology of comparison phrases.

and the *objects being compared* dimension into one dimension. The four dimensions of the simplified typology were therefore: (1) objects being compared and explicitness of referent; (2) difference quantified or not; (3) ordering of superiority; (4) language. The categories from the English typology that were used in the simplified typology are indicated with an asterisk in Table 5.4.

Because it was not possible to include four dimensions of variation in a two dimensional table, two simplified typologies were generated, one for the use of adjectives (Table 5.5) and one for the use of *-ngaphezulu* (Table 5.6). The dimension of language was included through the use of a simplified interlinear morphemic glossing, as described by Edmonds-Wathen (2019), thus including both the isiXhosa and the English phrase for each example. This made it easier to notice linguistic differences because, for the example in each cell, all other dimensions are fixed and only the language is varied.

Table 5.5: Example space of isiXhosa comparison phrases (adjectives)

Category	Adjective	Difference not quantified	Difference quantified
Implicit referent (objects)	*-nye* 'another'	Zoba ezinye izangqa. *zoba ezinye izangqa* draw some.other circles 'Draw more circles.'	Zoba ezinye izangqa ezintathu. *zoba ezinye izangqa ezi-ntathu* draw some.other circles that.are-three 'Draw three more circles.'
Explicit referent (objects)	*-ninzi* 'many'	Amapere maninzi kuneebhanana. *amapere ma-ninzi kunee-bhanana* pears are-many compared.to-bananas 'There are more pears than bananas.'	Amapere maninzi kuneebhanana ngamathathu. *amapere ma-ninzi kunee-bhanana nga-mathathu* pears are-many compared.to-bananas by-three 'There are three more pears than bananas.'
Explicit referent (numbers)	*-khulu* 'big'	Usibhozo mkhulu kunontlanu. *usibhozo m-khulu kuno-ntlanu* eight is-big compared.to-five 'Eight is bigger than five.'	Usibhozo mkhulu kunontlanu ngontathu. *usibhozo m-khulu kuno-ntlanu ngo-ntathu* eight is-big compared.to-five by-three 'Eight is three bigger than five.'

Source: Paper 2, Table 6

Table 5.6: Example space of isiXhosa comparison phrases (-ngaphezulu)

Category	Difference not quantified	Difference quantified
Implicit referent (objects)	Zoba izangqa ngaphezulu. *zoba izangqa ngaphezulu* draw circles above 'Draw circles above.'	Zoba izangqa ezingaphezulu ngezintathu. *zoba izangqa ezi-ngaphezulu nge-zintathu* draw circles that.are-more by-three 'Draw three more circles.'
Explicit referent (objects)	Amapere angaphezulu kuneebhanana. *amapere a-ngaphezulu* ***kunee****-bhanana* pears are-more compared.to-bananas 'There are more pears than bananas.'	Amapere angaphezulu kuneebhanana ngamathathu. *amapere a-ngaphezulu* ***kunee****-bhanana nga-mathathu* pears are-more compared.to-bananas by-three 'There are three more pears than bananas.'
	Amapere angaphezulu kweebhanana. *amapere a-ngaphezulu* ***kwee****-bhanana* pears are-above in.relation.to-bananas 'The pears are above the bananas.'	
Explicit referent (numbers)	Usibhozo ungaphezulu kunontlanu. *usibhozo u-ngaphezulu kuno-ntlanu* eight is-more compared.to-five 'Eight is more than five.'	Usibhozo ungaphezulu kunontlanu ngontathu. *usibhozo u-ngaphezulu kuno-ntlanu ngo-ntathu* eight is-more compared.to-five by-three 'Eight is three more than five.'

Source: Paper 2, Table 7

This paper demonstrates, among other things, what can be revealed by comparing languages in terms of a specific mathematical idea. In doing so, the paper contributes to the work on language-based differences in how mathematics ideas are expressed as described by Edmonds-Wathen (2019). In this case the comparison revealed the additional dimension of "how the ordering of superiority is expressed". The implications of this and other differences between the way in which isiXhosa and English express comparison phrases will be discussed in more detail in the following section.

5.2.3 Paper 3: Distribution of additive relation word problems in South African early grade mathematics workbooks

The third paper considers additive relation word problems and sets out two aims, rather than two research questions. These are to:

- expound on the existing literature to propose a comprehensive additive relation word problem typology; and
- analyse the prevalence of particular word problem types in the Foundation Phase DBE mathematics workbooks.

The unit of analysis in the third paper was the collection of additive relation word problems in the DBE workbooks. In this paper, the dimensions of variation and corresponding example space or typology that were used were not generated from the available examples, but were based on the existing literature. Drawing on previous typologies of additive relation word problems (particularly on the work of Carpenter, Hiebert and Moser (1981); Riley, Greeno and Heller (1984); and Roberts (2016a)), a comprehensive additive relation word problem typology, or example space, was set up (see Table 5.7). The comprehensive typology incorporated differences between the two "classic" typologies by differentiating between "ownership" and "attribute" categories for the collection category. It also reinstated the "equalise" category that was included in the early work, but that was later dropped (see for example Carpenter et al., 1988).

Table 5.7: Example space of additive relation word problems

Action types (movie)		
Subcategory	Change	
	Increase	Decrease
Result unknown	Nosisi had 3 marbles. Then she got 5 more marbles. How many marbles does she have now?	Nosis had 8 marbles. Then she lost 5 marbles. How many marbles does she have now?
Change unknown	Nosisi had 3 marbles. Then she got **some** more marbles. Now Nosisi has 8 marbles. How many marbles did she get?	Nosisi had 8 marbles. Then she lost **some** marbles. Now Nosisi has 3 marbles. How many marbles did she lose?
Start unknown	Nosisi had **some** marbles. Then she got 5 more marbles. Now she has 8 marbles. How many marbles did she have at first?	Nosisi had **some** marbles. Then she lost 5 marbles. Now she has 3 marbles. How many marbles did she have at first?
Subcategory	Equalise	
	Increase	Decrease
Change unknown	Nosisi has 3 marbles. Silo has 8 marbles. How many marbles does Nosisi have to win to have the same number of marbles as Silo?	Nosisi has 8 marbles. Silo has 3 marbles. How many marbles does Nosisi have to lose to have the same number of marbles as Silo?
Target unknown	Nosisi has 3 marbles. If she wins 5 marbles she will have the same number of marbles as Silo. How many marbles does Silo have?	Nosisi has 8 marbles. If she loses 5 marbles she will have the same number of marbles as Silo. How many marbles does Silo have?
Start unknown	Silo has 8 marbles. If Nosisi wins 5 marbles she will have the same number of marbles as Silo. How many marbles does Nosisi have?	Silo has 3 marbles. If Nosisi loses 5 marbles she will have the same number of marbles as Silo. How many marbles does Nosisi have?
Static types (photo)		
Subcategory	Collection	
	Different attributes	Different ownership
Collection unknown	Nosisi has 3 red marbles and 5 blue marbles. How many marbles does he have altogether?	Nosisi has 3 marbles. Silo has 5 marbles. How many marbles do they have altogether?
Subset unknown	Nosisi has 8 marbles. 3 are red and the rest are blue. How many blue marbles does he have?	Nosisi and Silo have 8 marbles altogether. Nosisi has 3 marbles. How many marbles does Silo have?
Subcategory	Compare	
	More than	Fewer than
Difference unknown	Nosisi has 8 marbles. Silo has 3 marbles. How many marbles does Nosisi have **more than Silo**?	Nosisi has 3 marbles. Silo has 8 marbles. How many marbles does Nosisi have **fewer than Silo**?
Compared quantity unknown	Silo has 3 marbles. Nosisi has 5 more marbles than Silo. How many marbles does Nosisi have?	Silo has 8 marbles. Nosisi has 5 fewer marbles than Silo. How many marbles does Nosisi have?
Referent unknown	Nosisi has 8 marbles. She has 5 more marbles than Silo. How many marbles does Silo have?	Nosisi has 3 marbles. She has 5 fewer marbles than Silo. How many marbles does Silo have?

Source: Paper 3, Figure 9

Like any phenomenon, additive relation word problems are composed of a number of different dimensions of variation, some of which are essential and some which are not (Runesson &

Kullberg, 2010). The dimensions of variation of additive relation word problems include: the three numbers in the additive relation; the names of the people in the problem; the objects in the problem; the number of sets in the problem; whether the problem situation is static or dynamic (see Figure 2 in Paper 3); the position of the unknown; and the direction of change (e.g. increase or decrease).

In order for learners to understand whether a particular aspect is essential or not, the dimension corresponding to that aspect needs to be varied while all the other dimensions are fixed. For example, to understand that the names of the people in the problem are not an essential aspect, learners need to see different problems which are the same in terms of all aspects except the names of the people in the problem. Roberts (2016b) has demonstrated the value of this approach when working with Foundation Phase learners. The typologies used in the literature and the comprehensive typology proposed in Paper 3 (Distribution of word problems) only include the essential dimensions of variation and keep the other dimensions fixed. In other words, all problems use the same three numbers, the same objects and the same names (see Table 5.7).

Table 5.8: Distribution of additive relation word problems in DBE workbooks

Subcategories	Gr 1	Gr 2	Gr 3	Gr 1	Gr 2	Gr 3
	Change					
	Increase			**Decrease**		
Result unknown	7	4	4	5	7	0
Change unknown	0	0	8	2	0	3
Start unknown	0	0	0	0	0	0
	Equalise					
	Increase			**Decrease**		
Change unknown	0	0	0	0	0	0
Target unknown	0	0	0	0	0	0
Referent unknown	0	0	0	0	0	0
	Collection					
	Attributes			**Ownership**		
Collection unknown	1	0	2	6	3	1
Subset unknown	1	4	0	0	0	0
	Compare					
	More than			**Fewer than**		
Difference unknown	1	0	1	0	0	0
Compared quantity unknown	0	0	0	0	0	1
Referent unknown	0	0	0	0	0	0

Source: Paper 3, Figure 13

Once the comprehensive typology was set up, it was possible to identify all examples of additive relation word problems in the canonical texts (61 examples) and to classify them according to the typology (see Table 5.8). This in turn made it possible to identify types of additive relation word problems that were under represented in the DBE workbooks, or not represented at all.

Paper 3 (Distribution of word problems) is the only paper in which language is not considered as a dimension of variation because only English examples of word problems were considered. However, the typology of Paper 3 (Distribution of word problems) was used as a basis for the typologies set up in Paper 4 (Compare type word problems).

5.2.4 Paper 4: Relative difficulty of comparison word problems: Learning from the case of isiXhosa

The fourth paper considers comparison questions and sets out to answer the following two questions:

- Are certain types of isiXhosa compare type problems easier for learners to solve than others?
- If so, is the relative difficulty influenced by:
    - the formulation of the comparative question?
    - the problem situation?

This paper in particular demonstrates the value of the theoretical framework and methodological approach outlined in Chapter 3. In the first phase of the research, the relative difficultly of three examples of difference unknown [more than] compare type word problems was established by means of an early grade mathematics assessment administered to Grade 1-3 isiXhosa learners in five rural Eastern Cape schools.

The three examples were taken from the isiXhosa version of the South African EGMA, the isiXhosa examples identified in Paper 2 (Comparison phrases), as well as from spoken language used during isiXhosa numeracy clubs. By identifying these three isiXhosa examples, Paper 4 (Compare type word problems) extends the work on different types of compare type problems. This work has previously primarily considered variations in English compare type problems, for example Hudson (1980) who introduced the “how many won’t get?” category and Roberts (2016b) who introduced the “how many missing?” category.

The comparison of the three problems confirmed that in isiXhosa, as in English, certain formulations of compare type problems are easier for learners to solve than others. However, drawing on the notion of different dimensions of variation, it became clear that the three examples used in the first phase, as well as those introduced by Hudson (1980) and Roberts (2016b), differed in terms of more than one dimension of variation and it was therefore not possible to establish the extent to which each dimension influenced the relative difficulty of the three word problems. In order to isolate the effect of the different dimensions, an example space or typology of compare type word problems was set up (see Table 5.9). While an English example space was also set up (see Table 5.10), the example space of compare type problems in Paper 4 (Compare type word problems) was originally set up and populated in isiXhosa.

Table 5.9: Example space of isiXhosa compare type word problems

		Problem situation		
		No matching	**Matching**	
			1-to-1 embedded	**1-to-1 not embedded**
Comparative Questions	**Classic**	Intombazana ineeswiti ezilithoba. Inkwenkwe ineeswiti ezine.	Umama uneembiza ezisibhozo neziciko ezihlanu.	Umama unabantwana abasixhenxe, abe neeorenji ezimbini.
	ngaphezu(lu) + -ngaphi?	Zingaphi iiswiti zentombazana ngaphezu kwezenkwenkwe?	Zingaphi iimbiza ngaphezu kweziciko?	Bangaphi abantwana ngaphezu kweeorenji?
	-ninzi + nga + -ngaphi?	Iiswiti zentombazana zininzi ngeeswiti ezingaphi kwezenkwenkwe?*	Iimbiza zininzi ngezingaphi kuneziciko?*	Abantwana baninzi ngezingaphi kuneeorenji?
	-ninzi + kangakanani?	Zininzi kangakanani iiswiti zentombazana kunezenkwenkwe?*	Zininzi kangakanani iimbiza kuneziciko?	Baninzi kangakanani abantwana kuneeorenji?
	Alternative	**No matching**	**1-to-1 embedded**	**1-to-1 imposed**
	ushota + -ngaphi?	n/a	Ushota ngeziciko ezingaphi?*	Ushota ngeorenji ezingaphi ukuze umama akwazi ukunika umntwana ngamnye iorenji enye?
	kusafuneka + -ngaphi?		Kusafuneka iziciko ezingaphi?	Kusafuneka iiorenji ezingaphi ukuze umama akwazi ukunika umntwana ngamnye iorenji enye?*
	ngazukufumana + -ngaphi?		Zingaphi iimbiza ezingazukufumana iziciko?	Bangaphi abantwana abangazukufumana ukuba umama unika umtwana ngamnye iorenji enye?

Source: Paper 4, Appendix B

In line with the study’s cross-cutting methodological approach, the first step in setting up the typology was to collect the examples of comparison problems from canonical texts. For Paper 4 (Compare type word problems), these examples were primarily drawn from the isiXhosa and English examples collected in Paper 2. While Paper 2 focused on comparison phrases, both those that were part of compare type word problems and those part of other phrases, Paper 4 (Compare type word problems) focused only on compare type word problems and included the problem context, not only the comparative question.

Table 5.10: Example space of English compare type word problems

		Problem situation		
		No matching	**Matching**	
			1-to-1 embedded	**1-to-1 not embedded**
Comparative Questions	**Classic**	A girl has seven sweets. A boy has five sweets.	A mother has eight pots and five lids.	A mother has seven children and two oranges.
	how many more?	How many more sweets does the girl have than the boy?	How many more pots than lids?	How many more children than oranges?
	Alternative	**No matching**	**1-to-1 embedded**	**1-to-1 imposed**
	how many missing?		How many lids are missing?	How many oranges are missing if the mother wants to give each child one orange?
	how many still needed?	n/a	How many lids are still needed?	How many oranges are still needed if the mother wants to give each child one orange?
	how many won't get?		How many pots won't get a lid?	How many children won't get an orange if the mother gives each child one orange?

Source: Paper 4, Appendix A

Once the set of compare type problems had been collected, two dimensions of variation with associated ranges of change were identified. The first dimension of variation was the problem situation. The range of change for this dimension included three cases: no matching, matching 1-to-1 embedded and matching 1-to-1 matching not embedded. The second dimension of variation was the phrasing of the comparative question. For the isiXhosa, there were a total of six options in the range of change in this dimension. These six options were grouped into classically phrased questions (three different phrasings) and alternatively phrased questions (three different phrasings) as seen in Table 5.9.

Once the different dimensions of variation and the corresponding range of change were identified and the structure of the example space was set up, the next step was to populate the space. While some of the categories in the example space could be populated from the available examples, thus determining the "current canonical example space", in order to set up the "potential canonical example space" missing examples had to be generated. In order to generate these I relied on isiXhosa-speakers to help formulate the examples.

Because the example space included many examples and because the examples were relatively long, rather than using interlinear morphemic glossing in each cell, as was done in Paper 2 (Comparison phrases), a separate table was set up for the English example space (see Table 5.10).

In the second phase of Paper 4 (Compare type word problems), four examples (Q1, Q2, Q3 and Q4 in Table 5.11) were compared in order to establish the effect of the "problem situation" dimension (Q1 vs Q3) and the "comparative question" dimension (both Q1 vs Q4 and Q2 vs Q3). The results of the comparison of the different compare type problems are discussed in more detail in the following section but once again the results confirmed that certain formulations of compare type word problems are easier for learners to solve than other formulations.

Table 5.11: Subset of isiXhosa compare type word problem example space

			Problem situation	
			No matching	**Matching (1-to-1 embed.)**
Comparative Question	*-ninzi* + *nga* + *-ngaphi?*	many + by + how many?	Q1	Q3
	-ninzi + *kangakanani?*	many + to what extent?	Q4	
	-shota + *-ngaphi?*	'short' + how many?		Q2

Source: Paper 4, Table 9

Although Paper 4 (Compare type word problems) did not directly compare English and isiXhosa comparison word problems, the two typologies or example spaces in Table 5.9 and 5.10 provide an opportunity to view language as a dimension of variation. By "lining up" the two tables (see Figure 5.3), and considering the corresponding cells in each table, it is possible to see the differences in language across a number of examples. Figure 5.3 is a specific instance of Figure 5.2 and shows the three dimensions of variation in Paper 4 and the range of change for each dimension.

By contrasting the two example spaces and their dimensions of variation and range of change, it is possible to see the influence of language on the manner in which compare type word problems are expressed. For example, it is possible to see that in English the range of possible change in these dimensions is more limited: there is only one way of phrasing the classic comparison question; there is only one way of expressing an ordering of superiority; and there is no variation in word order that is permitted. So, by extending the example space to include the dimension of language, it was possible to identify dimensions that could vary in some languages but do not vary in all languages.

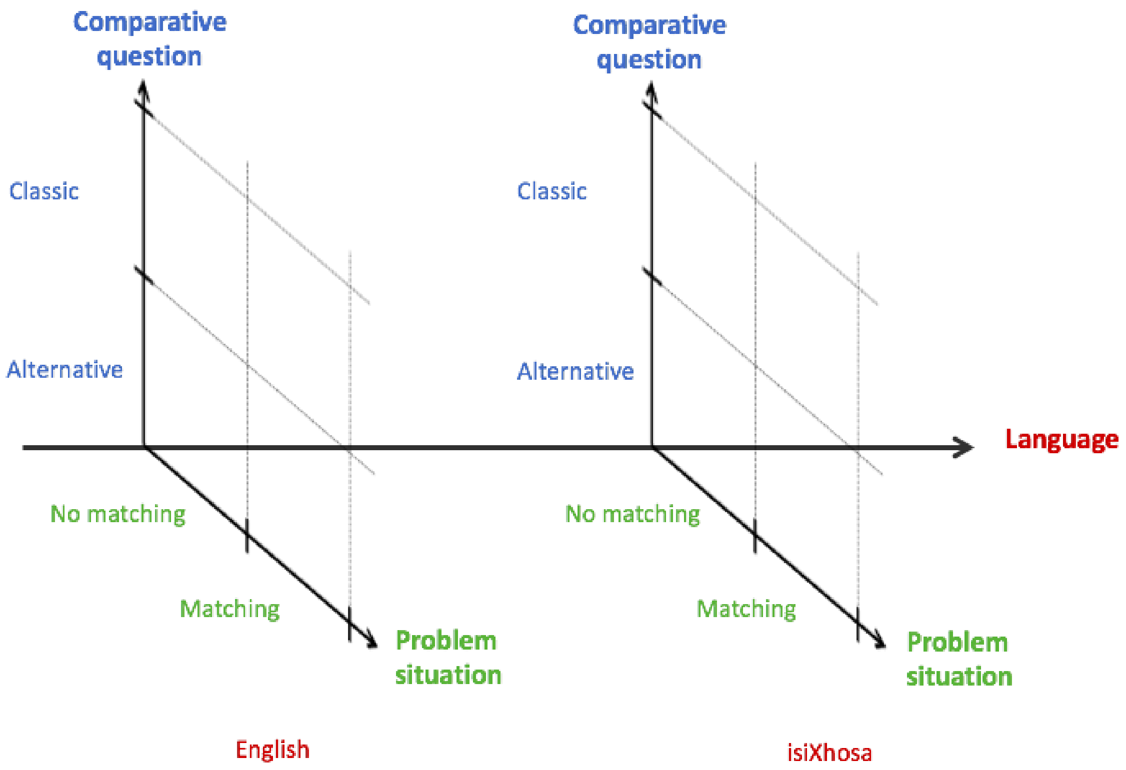


Figure 5.3: Structure of example space of English and isiXhosa compare type word problems

5.3 Answering the central research question

In the previous section I discussed the manner and extent to which each paper employed the theoretical framework and methodological approach outlined in Chapter 4. In this section I return to the central question and consider the way in which each paper contributed to answering the question:

> *What are the affordances and constraints of teaching early grade mathematics in rural isiXhosa-dominant classrooms, both in relation to the linguistic features of isiXhosa and the instructional practices of teachers?*

5.3.1 Paper 1: Number names: Do they count?

Paper 1 (Number names) answered the central question by considering the constraints and affordances of the linguistic features of isiXhosa in relation to the use of number names in the teaching and learning of mathematics. In order to answer this question, an analysis of the isiXhosa language (in particular of number words) was undertaken. The linguistic features pertaining to number that have been identified in other languages (Edmonds-Wathen et al., 2016) were used as the starting point for the analysis.

In terms of affordances, the paper identified the regularity and transparency of isiXhosa number names as two affordances for teaching. While not as transparent as certain Asian languages (Sun & Bartolini Bussi, 2018), isiXhosa number names lend themselves to thinking about the number of hundreds, tens and ones. This can be leveraged as a tool for helping learners understand the decimal system and that two- and three-digit numbers are constructed from groups of tens and groups of one hundred. Unfortunately, these affordances appear not to be leveraged, both in isiXhosa and in other Bantu languages, such as Sepedi (Mdluli, 2017). While the affordances are primarily for isiXhosa-dominant classes, they can also be leveraged when teaching mathematics in English, both when teaching English-speaking learners and when teaching isiXhosa-speaking learners, by introducing learners to the more regular and transparent number-naming system in isiXhosa.

By considering the constraints and affordances of the number names of an African language, Paper 1 contributed to the work on number names which, in mathematics education, has traditionally focused on number names in Indo-European and Asian languages with much less attention given to African languages. For example, in the recent 23rd International Commission on Mathematical Instruction Study "Building the Foundation: Whole Numbers in the Primary Grades" (Bussi, 2018), only two pages of a total of 488 pages, were dedicated to discussing the case of number names in African countries.

In terms of constraints, Paper 1 identified the length of number names and the differences between spoken and written language as two constraints for teaching mathematics in isiXhosa. In order to mitigate these constraints, the paper recommended the use of English number names as loanwords in isiXhosa mathematics classrooms (not exclusively, but in certain cases) as well as the careful articulation of isiXhosa number names to ensure that learners do not only hear isiXhosa number names during whole class counting activities. By including length of number name and differences between spoken and written language as influencing number names, the paper extended the work by Edmonds-Wathen et al. (2016) who identified syntactic category, transparency and regularity as linguistic features of language that influence number names.

5.3.2 Paper 2: Diversity of mathematical expression: The language of comparison in English and isiXhosa early grade mathematics texts

Paper 2 (Comparison phrases) answered the central research question in relation to comparison phrases and the implications that the linguistic features of isiXhosa have on the learning and teaching of comparison of quantities both in isiXhosa and in English. In order to answer this question, the linguistic tools used to express comparison in English (based on the work by linguists such as Kennedy (2009)) and in isiXhosa (based on the work by linguists such as Oosthuysen (2016)) were analysed, and the affordances and constraints of these linguistic tools for teaching mathematics were considered.

Paper 2 (Comparison phrases) identified two main affordances in the way in which isiXhosa expresses comparison. The first affordance is that isiXhosa explicitly signals "the difference" with the prefix *nga-* while in English "the difference" has to be deduced from the word order. The second affordance is that, unlike English, the structure of isiXhosa comparison phrases is not affected by the objects (or numbers) being compared. This means that learners do not have to navigate a shift from comparing objects belonging to a person, to comparing numbers. Typically the work on how different languages express comparison only includes the way languages indicate the ordering of superiority/inferiority and the standard against which the comparison is made (Kennedy, 2009). This paper extends the work on how language express comparison by including two other components, namely the way in which a language signals "the difference" in a comparison phrase; and the way in which comparison phrases are structured.

In terms of constraints, Paper 2 (Comparison phrases) identified the fact that one of the isiXhosa words used to express an ordering of superiority, namely *-ngaphezulu*, can mean both 'more' and 'above'. This can cause a conflict when representations such as the standard one hundred chart are used where bigger numbers appear below smaller numbers. However, this constraint can be mitigated by using a bottom-up number chart where bigger numbers appear above smaller numbers. While Bay-Williams and Fletcher (2017) have made a similar argument for English where "more" is associated with "higher", the argument is stronger in isiXhosa where the word used for "higher" is the same as one of the words used for "more".

A second observation is that, as explained previously, isiXhosa uses different adjectives as well as the adverb *-ngaphezulu* 'more/above' to indicate an ordering of superiority while English only uses the word *more* (except in the case of numbers where *bigger*, *larger*" and *greater* are also used). It is not clear whether this is a constraint because it means that learners need to learn more words and need to know when to use which words, or whether it is an affordance as learners are able to access the same concept through different words.

5.3.3 Paper 3: Distribution of additive relation word problems in South African early grade mathematics workbooks

Unlike the other three papers, Paper 3 (Distribution of word problems) focuses more on the implications of the instructional practices of isiXhosa-dominant classrooms than on the linguistic features of isiXhosa. However, in as much as it considers additive relation word problems, it also lays the foundation for the work in Paper 4 (Comparison questions) on the influence of linguistic features of isiXhosa as they relate to compare type additive relation word problems.

Paper 3 (Distribution of word problems) answers the central research question by considering the affordances and constraints, in relation to learners' exposure to additive relation word problems, of the heavy reliance by teachers in rural isiXhosa-dominant classrooms on DBE workbooks. In order to answer the question, the frequency of different types of additive relation word problems was established, using a comprehensive typology of additive relation word problems. The results of this analysis showed that the DBE workbooks only contain a small number of additive relation word problems, namely 61, and that there is an uneven distribution of additive relation word problems. Of the 61 word problems, there were 40 change type problems, 18 collection type problems, 3 compare type problems and no equalise problems. Therefore, the biggest constraint of an over reliance on DBE workbooks is the lack of exposure that learners have to the full range of additive relation word problems.

The observation of the heavy reliance on DBE workbooks that prompted Paper 3 (Distribution of word problems), confirms Taylor's (2013) observation that workbooks are used for more than supplementary worksheets. The results of Paper 3 provide an additional reason why it is problematic that workbooks are used as a primary teaching resource, namely that they do not provide enough exposure to all the types of additive relation word problems. This contributes to the findings of previous studies which have identified a lack of conceptual signaling (Hoadley & Galant, 2016); and misconceptions in relation to addition (Davis, 2012) and to multiplication (Galant, 2013) as reasons why it is problematic that teachers rely primarily on the DBE workbooks.

Although this study is located within rural isiXhosa-dominant schools which use the isiXhosa version of the DBE workbooks, in this paper the English version of the workbooks were analysed. The first reason for this is that it was easier for me to work with the English examples. The second reason is that because the isiXhosa workbooks are a direct translation of the English workbooks, the frequency of word problems in the isiXhosa workbooks is the same as the frequency of word problems in the English workbooks, and therefore the examples from either workbook could have been analysed.

While an over reliance on instructional materials provided by the DBE can be a constraint when there are problems with the materials, such as a lack of sufficient variation in the types of word problems, the over reliance can also be an affordance if the materials that teachers are provided with are well designed and appropriate for the classrooms in the dominant school system.

5.3.4 Paper 4: Relative difficulty of comparison word problems: Learning from the case of isiXhosa

Paper 4 (Compare type word problems) answers the central research question in relation to compare type word problems. The affordances and constraints of the different ways in which isiXhosa expresses compare type word problems were discussed in relation to the relative difficulty of the different compare type word problems, as measured by the percentage of learners who correctly solved each problem in an early grade mathematics assessment. The analysis revealed two important affordances in terms of isiXhosa linguistic tools used to express compare type word problems, in particular in terms of how to express comparative questions.

The first affordance is that isiXhosa has a specialised question word *kangakanani* 'to what extent' which can be used to ask "how many more" questions. In English, learners often confuse "how many" and "how many more" questions (Roberts, 2016b). In isiXhosa "how many" questions are asked using the question word--*ngaphi.* Because *kangakanani* is different to -*ngaphi,* it is possible that isiXhosa learners will be less likely to confuse *kangakanani* 'how many more' questions with -*ngaphi* 'how many' questions. This hypothesis is supported by the results from Paper 4 (Compare type word problems) which show that more learners correctly solve "how many more" problems when the question is phrased using *kangakanani* 'to what extent' as opposed to -*ngaphi* and -*ninzi* ('how many' and 'many').

The second affordance is the loan word -*shota* ‘to be short of’ (Oosthuysen, 2016). While English typically uses ‘to be short of’ in very specific contexts, in isiXhosa -*shota* can be used when comparing the numerosity of any two sets that have some reason to be matched. The results of Paper 4 show that more learners correctly solve compare type word problems when the question is phrased using -*shota* than when -*ngaphi* and -*ninzi* (‘how many’ and ‘many’) are used. These are two important affordances that are available in isiXhosa which are not available in English. By not leveraging these affordances, the full effect of home language instruction is unlikely to be realized.

In terms of constraints, the collection of comparative questions as they appear in the canonical texts include examples of incorrect translations of “how many more” questions. The canonical texts also do not include any explicit or systematic mention of the different ways in which “how many more” questions can be phrased or of the affordances and constraints of the different phrasings. This points to an implicit emphasis on vocabulary rather than on the mathematics register (Halliday, 1978). As previously pointed out in Chapter 2, there is a difference between knowing words such as *more* and *less* and learning the “language patterns” associated with the words and how the words are used to construct mathematical concepts (Schleppegrell, 2007, p. 143). Paper 4 suggests that the same is true for words such as “how many” and “how many more” – children need to learn the language patterns associated with these words, not only the words themselves.

5.4 Contributions of the whole study

In the previous sections, I discussed the way in which each paper employed the theoretical and methodological perspectives set out in Chapter 4 and the way in which each paper answered the central question. In this section I consider the contributions of the collection of the papers as a whole. These include (1) the use of “partial embedding” as a methodological approach for working in rural schools, (2) the use of examples from canonical texts as unit of analysis, (3) the extension of the notion of a dimension of variation to include language, (4) the use of specific tables as an analysis tool and (5) the use of IMG in mathematics education research.

5.4.1 Partial embedding as a methodological approach

The first contribution of the study as whole is that it provides an example of how mathematics education specialists who are not first language speakers might work together with first language speakers and linguists to understand the linguistic features of a particular language, for example isiXhosa, and to consider the implications that these features could have on learning and teaching mathematics. This approach draws on similar approaches used by other mathematics education researchers, for example Borden (2010, 2011, 2013) who studied Mi’kmaq and Edmonds-Wathen (2012, 2014, 2019) who studied Iwaidja.

While the overarching methodological approach is not emphasised in the individual papers, it has been discussed in the introductory chapters. Both of the important features of this approach involve time. The first feature is the need for the researcher to spend time gaining a functional understanding of the language and to spend time seeking opportunities to hear the language being used to express mathematical ideas. This can be done by, for example, spending time in mathematics classrooms, or, in the case of Borden (2010, 2011, 2013), by teaching in a school where the language she studied was the home language of the learners.

The second feature is the need to work closely with language specialists. There are a number of early grade mathematics education specialists who are fluent in languages that differ significantly from Indo-European languages, for example Feza (2016) who is fluent in isiXhosa, Mdluli (2017) who is fluent in Sepedi, and Trinick (2015) who is fluent in Māori. These scholars bring extremely valuable knowledge to the education knowledge project and are able to engage in research which considers the effect of the linguistic features of the language they speak on the learning and teaching of mathematics in that language. However, when a mathematics education specialist, like myself, is not fluent in the language they are studying, they need to work closely with language specialists in order to do similar work. Both Borden (2013) and Edmonds-Wathen (2019) used this approach.

As someone who is not proficient in the language being studied, the process of trying to understand how the language is used to express mathematical ideas by consulting different people who provide different and sometimes contradictory examples, is a long and iterative process. This is particularly true of Paper 2 (Comparison phrases) which was conceptualised in June 2018 but was only finalized in July 2020. Throughout this period, I was revising my understanding of comparison phrases, revisiting the same examples many times as different people provided different insights. Even so, it will be unsurprising if the paper still contains mistakes, or at least examples of translations or expressions that will not be agreed on by all.

5.4.2 Canonical texts as a unit of analysis

A second contribution of the study as a whole is the positioning of examples from canonical texts as objects worth studying. This was done by extending the idea of an example space (Watson & Mason, 2005) to include examples of linguistically expressed mathematical ideas found in canonical texts. In each paper a different set of examples was the unit of analysis: number names, comparison phrases, word problems in CAPS, DBE, NMI and NECT workbooks and EGMA assessments. Together the four papers showed that canonical texts are a valuable object of study.

In the four papers, examples from canonical texts were used as a unit of analysis in three different ways. Firstly, in Paper 3 (Distribution of word problems) the examples were used to establish the extent to which learners are likely to encounter a particular topic or type of question in a particular grade. This use of examples in canonical texts extends the work of numerous other studies on the distribution of a particular type of problem in textbooks. Examples of countries where similar studies considering the distribution of word problems in textbooks have been done include the United States of America and the then Soviet Union (Stigler et al., 1986), Turkey (Olkun & Toluk, 2006; Tarim, 2017) and Brunei (Dhindsa et al., 2014).

Secondly, in Papers 1, 2 and 4 the examples in the canonical texts were used as a starting point for establishing the different ways in which a language whose mathematics register is still being developed, expresses mathematical ideas. As mentioned previously, the examples in the canonical texts are not necessarily the best way of expressing mathematical ideas in a particular language, for example isiXhosa. Rather, the examples provide a concrete object for linguistics, mathematics education specialists and teachers to engage with in order to answer questions such as:

- Of all the different ways a mathematical idea could be expressed in isiXhosa, is there one that is most helpful for learners to be able to understand the mathematical idea?
- Which of all the possible ways of expressing a mathematical idea should learners be exposed to? Should they be exposed to all, some, or only one?
- If learners are to be exposed to more than one way of expressing a mathematical idea, in which order should they be exposed to the different ways? All at once, or one by one?

In this sense the examples provide a means to support the kind of work that has been done in other countries to develop the mathematics register of indigenous languages, in particular the work done in New Zealand to develop a mathematics register for Māori, as described by Barton et al. (1998).

Thirdly, again in Papers 1, 2 and 4, the examples from the canonical texts are used to compare the way in which different languages express mathematical ideas. There are studies that have used mathematics texts, particularly textbooks, to identify differences in how mathematical ideas are presented in different countries. For example, Li (2000) compared American and Chinese mathematics textbooks in terms of problems presenting the addition and subtraction of integers. However, to the best of my knowledge, no other studies have used the canonical texts from one country, available in multiple languages, as a unit of analysis. The fact that South Africa has invested in translating mathematical texts into multiple languages, provides a useful data set for language in mathematics studies. This third use of examples from canonical texts as a unit of analysis leads to the next contribution of the study as a whole, namely the notion of language as a dimension of variation.

5.4.3 Language as a dimension of variation

A third contribution of the study as a whole is the extension of the notion of a dimension of variation from variation theory (Marton & Booth, 1997) to include the dimension of language when studying canonical texts. As explained in Chapter 3, considering the language in which a mathematical idea is expressed as a dimension of possible variation deepens reflection on the mathematics. Therefore, having examples of languages which have different linguistic structures, such as isiXhosa and English, is a powerful resource.

In Paper 1 (Number names), Paper 2 (Comparison phrases) and Paper 4 (Compare type word problems) I demonstrate what considering language as a dimension of variation when studying mathematical ideas could look like. By comparing English and isiXhosa number names in Paper 1, the constraints and affordances of each languages' number-naming conventions were made clear. As mentioned previously, Paper 1 extended the work on number names which typically focuses on Indo-European and Asian languages (Bussi, 2018). The long yet transparent isiXhosa number names were contrasted with the shorter but non transparent and irregular English number names.

The differences in the types of words used for number names were also considered. While both English and isiXhosa use numbers as nouns and as adjectives, in isiXhosa different forms of number names are used when numbers are used as nouns and when they are used as adjectives. This provides insights both into English and isiXhosa grammar as well as into the different ways in which languages use linguistic tools to express mathematical ideas (Edmonds-Wathen, 2019). While other research has considered the different types of words used by different languages to express numbers (e.g. verbs in both Mi'kmaq (Borden, 2011) and in Māori (Barton, 2008)), this study has extended this work by foregrounding the way in which one language, isiXhosa, uses different forms of numbers when using them as nouns and when using them as adjectives.

Similarly, by comparing the way in which isiXhosa and English express comparison, a number of important ideas about comparison emerged. For example, the importance of "the difference" in a comparison, and that this can either be indicated by the word order (in English) or by a prefix (in isiXhosa) and what potential influence this has on learning mathematics. By varying the dimension of language, while keeping the mathematical ideas invariant, it was possible to see these aspects of comparison phrases. This in turn gives us some more insights into how mathematics is constructed in different languages.

5.4.4 'Example space' tables

A fourth contribution are the "example space" tables – both the process of creating the tables and the tables themselves. The process of generating the structure of the table based on examples in canonical texts, and then populating this table, first with examples available in the texts and then by creating the missing examples can be seen in Paper 2 (Comparison phrases) where the missing examples are in italics in Table 5.4. This process is also seen in Paper 3 (Distribution of word problems) where the "missing" equalising problems were reintroduced (see Table 5.7), as well as in Paper 4 (Compare type word problems) where only the examples with asterisks in Table 5.9 were in the EGMA tests and the other examples were generated afterwards.

These tables provide teachers, teacher trainers and material designers with a reference resource. For example, Paper 2 (Comparison phrases) provides a categorisation of all the different types of comparison phrases Foundation Phase learners are expected to be familiar with. This can be used by teachers and materials designers to ensure learners are exposed to all the different types of phrases. As such, the table can be used as a tool to help ensure that learners do not only know mathematical words such as *more*, *less*, and *as many as* but that they also "learn the language patterns associated with these words and how they construct concepts in mathematics" (Schleppegrell, 2007, p. 143).

Similarly, Paper 3 (Distribution of word problems) provides a typology of additive relation word problems which teachers and materials designers can use to ensure learners are exposed to all the different types of word problems. Finally, Paper 4 (Compare type word problems) provides a table of all the different ways in which comparison questions can be asked in isiXhosa, again providing teachers and materials designers with a resource to use when deciding which questions learners should be introduced to and in which order.

5.4.5 Interlinear morphemic glossing as a means of presenting "unknown" language

Finally, this study makes a contribution in terms of exemplifying how interlinear morphemic glossing (IMG) can be used in mathematics education research. Building on the examples given by Edmonds-Wathen (2019) in her paper on *Linguistic methodologies for investigating and representing multiple languages in mathematics education research*, Paper 2 (Comparison phrases) and Paper 4 (Compare type word problems) in this study demonstrates how IMG can be used when comparing translations of canonical texts with the original language and the value of such a comparison.

As pointed out by Comrie et al. (2008) in Edmonds-Wathen (2019), the purpose of an IMG is "to give some further possibly relevant information on the structure of a text or an example, beyond the idiomatic translation" (p. 2). This is particularly important when the structure of the source language is very different to the structure of the target language, as is the case with isiXhosa and English. In Paper 2 (Comparison phrases), IMG was used to examine what linguistic tools English and isiXhosa use to indicate "the difference" when comparing objects in terms of numerosity. In English, the linguistic tool is word order, but in isiXhosa, the linguistic tool is the prefix *nga-*, as seen in example (3).

(3) Amapere maninzi kuneebhanana ngamathathu.

amapere	*ma-ninzi*	*kunee-bhanana*	*nga-mathathu*
pears	are-many	compared.to-bananas	by-three

'There are three more pears than bananas.'

Similarly, in Paper 4 (Compare type word problems), IMG was used to demonstrate the different ways in which it is possible to phrase classic compare questions in isiXhosa; in contrast to English in which only one phrasing is possible (see Table 5.12).

Table 5.12: Different formulations of the classic "how many more" comparative question in isiXhosa

Variation	**Example**
-ngaphi? + ngaphezu(lu)	Zingaphi iimbiza ngaphezulu kweziciko? *zi-ngaphi iimbiza ngaphezulu kwe-ziciko* they.are-how.many pots more compared.to-lids
-ninzi + nga- + -ngaphi?	Iimbiza zininzi ngezingaphi kuneziciko? *iimbiza zi-ninzi nge-zi-ngaphi kune-ziciko* pots they.are-many by-they.are-how.many compared.to-lids
-ninzi + kangakanani?	Zininzi kangakanani iimbiza kuneziciko? *zi-ninzi kangakanani iimbiza kune-ziciko* they.are-many to.what.extent pots compared.to-lids

Even though I did not use IMG in Paper 1 (Number names), in example (1) I demonstrated how its use could have strengthened the paper.

5.5 Limitations

Although this study has made a number of contributions, it also has a number of significant limitations, primarily because of my position as an outside specialist. The first major limitation is my lack of linguistic resources. The other is that even though I made an effort to be embedded in the rural context, this was only a partial embedding. I was still primarily embedded in the field of academic research with its limited views of who is an specialist, and what is considered an acceptable way of generating research.

5.5.1 Limited linguistic resources

Because I am not proficient in isiXhosa, I am limited in terms of the kind of research I am able to do concerning the use of isiXhosa to teach Foundation Phase mathematics. I am limited in terms of knowing what kinds of questions to ask, of having access to the full range of ways in which ideas can be expressed, and of understanding the implications of the differences in how ideas are expressed. While it is possible to establish these things through working closely with first language speakers, this work is still limited and bound to include mistakes. To exemplify these limitations, I discuss the particular limitations of the studies in Paper 1 (Number names) and Paper 4 (Compare type word problems).

In Paper 1 (Number names), I missed two important issues pertaining to the use of numbers names in isiXhosa classrooms. The first is the practice of using English number names as loanwords in isiXhosa. Like other loanwords, English numbers are used with isiXhosa prefixes, in the case of number names, they are used with the prefix *u-* from noun class 1, for example *u-two* (Futuse, 2018).

The second and related aspect of isiXhosa number names that I did not address is the difference between the "personified" and the "abstract" forms in the naming of numbers in isiXhosa, as used in mathematics classes and mathematics texts. While these two different ways of naming numbers are not used when counting, they are used when referring to relationships between numbers. It was therefore only in writing Paper 2 (Comparison phrases) and 4 that I became aware of this important difference. The difference is significant and isiXhosa speaking adults, especially teachers, are divided as to whether it is acceptable to refer to number names in the personified form, for example *untlanu* 'five', versus the more formalized form, for example *isihlanu* 'five', when describing the relationship between numbers.

Secondly, while working on Paper 4 (Compare type word problems), I only became aware of the full range of variation possible in how to ask comparison questions after the two assessments had already been administered. Although I was aware of the variation possible in terms of the question word *-ngaphi* versus *kangakanani* - I was not aware that *-ngaphi* could be used with two different ways of expressing "more", namely *-ninzi* and *-ngaphezulu*. I was also not, and still am not, in a position to understand the complexities of the contrasting positions relating to the use of the loanword *-shota*. On the one hand, there are those who argue that it should not be used in formal mathematical texts as it is a loanword from English. On the other hand, there are those who argue that it should be used as learners understand the meaning of the word and it helps them to understand the concept of difference.

The questions I failed to ask and the nuances I failed to pick up all emphasise the complexity of language research - which is compounded by the complexity of mathematical expressions, and by my lack of proficiency in language isiXhosa. In order to engage with these complexities, interdisciplinary research is necessary – such that can bring together knowledge of mathematics, early grade pedagogical content knowledge, the tacit knowledge of early grade mathematics teachers, as well as the knowledge of isiXhosa language specialists.

5.5.2 Only partially embedded in rural contexts

The second limitation I brought to the research is that I am not fully embedded in the context I was researching. While I was able to partially embed myself in the mainstream education system by spending many hours observing isiXhosa-dominant classrooms in rural schools, I am still primarily embedded in the privileged education system.

At the end of my research journey, I am again reminded of the realisation I experienced at its beginning – namely, that what I thought was appropriate research for me to do (a classroom-based design experiment) was not appropriate for a number of reasons as described in Chapter 1. One of these reasons was the minimal value that teachers I observed placed on word problems. I am again faced with a similar question: to what extent do each of my papers speak to the challenges and priorities of rural teachers? To what extent are the research questions I posed in each paper, questions that the teachers in rural schools would have chosen to research?

I acknowledge the extent to which my own research was informed by my outside observations of the linguistic features of isiXhosa, more than by the needs of the inside specialists, the teachers and coaches, of what is necessary to study. However, I also acknowledge that while my research might not address the immediate micro-level classroom needs of the teachers in the mainstream system, the research has the potential to make a longer term and macro scale contribution to improving the canonical texts used by both systems.

5.5.3 Limited framing of comparative language

After the publication of Paper 2, Edmonds-Wathen (2020) in personal communication, noted that the description of comparative language in general in Paper 2 referred only to "standard" type comparative language. It would have been preferable to consider a wider cross linguistic typology of comparative language. The "exceed" and "implicit" types of comparative language should have been mentioned, before focusing on the standard type of comparative language. In addition, drawing from linguistics, the quantification of comparison could have been referred to as the "degree", which would have facilitated more connection with the linguistic literature. This noted as a limitation of this research study, which also has implications for Paper 4. Future published work will ensure that this is corrected.

This limitation arises from the constraints imposed when working in a cross-disciplinary manner where peer researchers in mathematics education may not have the linguistic knowledge to critique that aspect of the paper.

5.6 Implications

This study has certain limitations, but it also has implications for a number of different aspects of education. These include implications for teaching in a multilingual context, for instructional design as well as implications for initial teacher training and further research.

5.6.1 Teaching in a multilingual context

While the four papers focused on the affordances and constraints of teaching mathematics in isiXhosa-dominant classrooms, some of the findings are also applicable for non-homogenous linguistic contexts, particularly contexts where the language of instruction is English, but not all learners are English-speaking. For example, Paper 2 (Comparison phrases) recommends that teachers be aware of the different ways in which language express comparison phrases so that they can leverage the affordances of different languages. English teachers could express comparison phrases in a way that makes "the difference" explicit: *Five is more than three by two*. English teachers could also initially construct comparison phrases which specify "the difference" using a direct translation of the isiXhosa phrasing, namely *five is more than eight by three.* This phrasing more closely aligned to the comparison phrase that does not specify the difference, namely *five is more than eight*, than the more commonly used phrasing of *five is three more than eight.*

Being aware of language differences also enables teachers to support learners learning mathematics in a second language, especially one with linguistic features that are significantly different to the linguistic features of their home language. For example, teachers could ensure that isiXhosa-speaking learners who are learning mathematics in English are exposed to the different word order used in English when comparing different things, namely "A is more *adj* than B" when comparing a property of two objects; and "There are more A than B" when comparing the quantity of two sets of objects. Teachers could also pay careful attention to the way in which English expresses an ordering of superiority, namely by using *-er* or *more*, which is significantly different to isiXhosa where an adjective or adverb together with *kuna-* 'compared to' is used to express an ordering of superiority. In this sense the papers contribute to the research necessary for language responsive teaching in mathematics (Prediger, 2019).

5.6.2 Initial teacher training

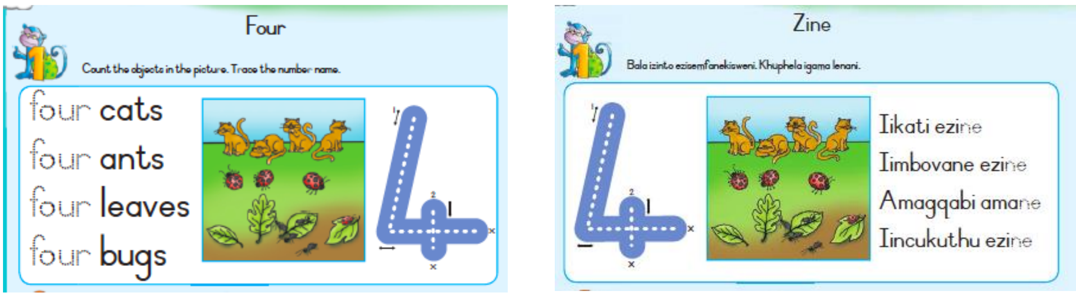
The study has implications for how universities structure and plan their initial teacher education courses both in terms of the language policy and in terms of the content. If the knowledge project of the South African education system is to serve the majority of learners, namely African language learners, it is important the students who will teach these learners are able to learn how to teach mathematics, at least to some extent, in the African language they will use to teach mathematics. Currently, as pointed out by Ramadiro and Porteus (2017) and as demonstrated by Alex et al. (2020) and Roberts and Alex (2020), many students studying to become early grade teachers are not proficient in the mathematics register of the African language that they will use to teach mathematics, even though this language is their home language. This study provides concrete examples of how the mathematics register of different languages differ, thereby emphasizing the importance of including instruction in teaching mathematics in the language that students will use to teach mathematics as part of initial teacher education.

In terms of what is taught (not what language it is taught in), the collection of papers in this study offers some examples of topics that could be covered. For example, the number naming conventions of different languages as well as the way in which different languages express mathematical ideas such as comparison. Although the papers are limited to isiXhosa, because of similarities between Bantu languages, the examples from isiXhosa can be used as a starting point for exploring examples from other languages.

5.6.3 Design or translation of instructional materials

In terms of the design of instructional materials, the study has implications for the way language is used, as well as for the order in which word problems are introduced. The fact that this study has implications for the way in which language is used in instructional materials is an example of what Edmonds-Wathen (2019) hypothesised, namely that studying mathematical language can lead to the documentation and description of languages which can then be used for "the production of teaching resources and the development of a mathematics register for teaching the mathematics in the language" (p. 132).

When instructional materials are translated into or designed for a particular language, it is important that the linguistic features of that language are taken into consideration. For example, if the difference in English and isiXhosa number naming systems are taken into consideration when designing instructional materials, the materials in the two languages might differ. IsiXhosa materials might leverage the transparent number names in ways that are not possible in English. IsiXhosa materials also might use different objects when introducing number names in order to take into consideration the influence of the noun class system on the form of the number names when referring to objects, as demonstrated in Figure 5.4.



Source: Paper 1, Figure 1

Figure 5.4: A comparison of "four" used as an adjective in the English and isiXhosa DBE workbooks in Grade 1 Term 1

Another consideration when designing isiXhosa instructional materials are the different forms of number names that are possible in isiXhosa, namely the "abstract", "adjectival", "personified" or English loanword form, each belonging to a particular class with a particular prefix, as exemplified in Table 5.13.

Table 5.13: Different forms of the isiXhosa number name for "eight"

Number name	Form	Class	Prefix
isibhozo	abstract	7	*isi-*
-sibhozo	adjectival	7	Depends on noun being described
usibhozo	personified	1	*u-*
u-eight	loanword	1	*u-*

The different forms (with their different prefixes) impact on the design of materials because mathematics materials typically use number symbols in sentences, rather than writing out the number names, and in isiXhosa, when number symbols are used in a sentence, the prefix is used with the number symbol, see example (4):

(4) u-8 ngaphezulu kune-3 ngu________.

u-8 *ngaphezulu* *kune-3* *ngu-________*
8 more compared.to-3 is-________

'8 more than 3 is ________.'

This means that in examples such as (4), even though the prefix does provide some information, it is not clear which form of the number name the author had in mind. This means that the teacher and learners might be using a different form of the number name than the one intended. Pimm (2018) raises a similar concern in relation to the use of mathematical symbols to present classroom transcripts pointing out that “there are significant differences between speaking and writing in relation to numbers” (p. 72). An analysis of the isiXhosa canonical texts also shows that there is no consistency in terms of when a particular prefix is used, and when a different prefix is used.

In terms of the content in instructional materials, Paper 3 (Distribution of word problems) demonstrates the importance of ensuring that examples of all types of word problems are well represented in materials. While others, for example the Common Core Standards Writing Team (2011), have called for a deliberate order in the introduction of additive relation word problems starting with the easiest (change type problems) and ending with the most difficult (compare type problems), Paper 4 (Compare type word problems) allows for an even more fine-tuned introduction by establishing the relative difficulty of different variations of compare type problems.

5.6.4 Further research

Finally, this study points to a number of potentially fruitful areas for future studies that aim to be accountable to African language-dominant mathematics classrooms. These will be discussed in terms of specific findings from the papers, and in terms of more general areas of study relating to the use of language in rural schools.

Firstly, it would be possible to study the aspects of Paper 1 (Number names) and Paper 4 (Compare type word problems) that were not included, due to my limited knowledge of isiXhosa. Building on Paper 1 (Number names), the use of English number names as loanwords both inside and outside the classroom could be studied, drawing on the work by Futuse (2018) who looked at loanwords in general, but included number names as a specific example of such loan words. Similarly, the different forms of number names used when comparing numbers, namely the “personified” and the “abstract” form, could be studied, as well as the constraints and affordances of each form.

Leading on from Paper 4 (Comparison questions), it would be possible to do a more systematic comparison of the different ways of phrasing comparison questions. This work could be strengthened by engaging with learners about how they understand each of the different ways of phrasing the comparison questions or what strategies they use to solve them.

Secondly, this research has demonstrated the value of identifying ideas in early grade mathematics that are particularly influenced by language. While this study focused on number names and comparison phrases, during my observations and conversations with teachers I identified the following topics that appear to be influenced by language: (1) the use of *before* and *after* to describe the relationship between numbers when their ordinality is considered; (2) the naming of fractions; (3) the language of multiplication and division; and (4) the language of measurement. Because of the original focus of this thesis on additive relation word problems, these topics were beyond the scope of this PhD, but will be pursued after its completion.

Thirdly, this research has demonstrated the rich source of examples, including incorrect examples, that are provided by canonical texts. As documents that heavily influence the enacted curriculum, canonical texts are worthy of further study. While this study has analysed canonical texts in terms of examples of linguistically expressed mathematical ideas, the texts also provide a source of examples of the way graphic representations such as number lines are used to express mathematical ideas. Analysing these representations would provide valuable insight into the commonly seen representations and would contribute to the domain of textbook analysis.

This study is situated within a particular context: a context in which early grade teachers and learners share an African language as a home language, which is also used as the language of instruction, but which the teachers are not necessarily proficient in using to teach mathematics. As I have pointed out previously, while this study seeks to speak to this particular context, it does not seek to describe or analyse the language use in these contexts. Describing and analysing the language used in *African language*-dominant Foundation Phase classes would allow a comparison between these classes and bilingual classes, both those with teachers who are proficient in teaching mathematics in the language of instruction and those who are not yet proficient. In particular, studies done within contexts where African languages are used to teach mathematics could aid the development of what Prediger et al. (2019) refer to as "intermediary language". An example of intermediary language is the use of *take away* as an intermediary term to describe the mathematical term *subtraction*. Such research would help to establish what intermediary language already exists and what needs to be developed to support isiXhosa learners in moving from everyday isiXhosa to the formal isiXhosa mathematics register.

5.7 Answering the overarching question

In the previous two sections I have outlined the contributions, limitations and implications of this study. Understanding these aspects of the study are necessary to answer the study's overarching research question:

> *What research possibilities are available for an "outside specialist" wishing to build the education knowledge project in ways that are accountable to rural isiXhosa-dominant schools?*

This study, and the four papers at its core, demonstrate both what an outside specialist conducting academic research can and cannot contribute to building the knowledge required in the mainstream education system. This research demonstrates that an outside specialist can:

- contribute an understanding of mathematics and mathematics education;
- identify ideas in mathematics that are worth studying in relation to language;
- ask questions that a first language speaker would not consider asking;
- engage, with help from others, with text in the language they are studying; and
- highlight the realities of learning and teaching on the periphery of the main system.

This research also demonstrates that an outside specialist cannot:

- do research that involves trialing a particular teaching approach;
- analyse the complexities of spoken language as used in a classroom;
- analyse the complexities of written language, for example, the differences in emphasis when word order is changed in a language with flexible word order; and
- do research that entails asking learners to talk about how they understand mathematical ideas.

Together this collection of papers provides one limited example of the kind of research that is aligned to the linguistic resources of the majority of children, and that contributes to realising the promise of mother tongue-based bilingual education (Ramadiro & Porteus, 2017). While there are many other problems that need to be addressed, building the education knowledge project through the kind of research demonstrated in this study is a necessary and important part of shifting the South African education system from a bimodal achievement distribution to a normal achievement distribution.

In particular, this collection of papers demonstrates the value of understanding the ways different languages express mathematical ideas. As demonstrated in the papers, the comparison of different languages in terms of specific mathematical ideas can provide insights into the nature of mathematics and can provide insights for learning and teaching mathematics in both languages. Taken as a whole, the contributions made by each paper help to answer what Edmonds-Wathen (2019) describes as the "many unanswered questions about the diversity of mathematical expression in different languages, such as the extent to which mathematical concepts vary across languages, and how similar concepts can be expressed in different languages" (p. 119).

As I have already laid out in detail in the introductory chapters, this kind of research would not have been possible without the many hours of time and the vast amounts of detailed knowledge of isiXhosa that so many individuals shared so generously with me.

Ndiyabulela.

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